Stochastic Lyapunov-Barrier Functions for Robust Probabilistic Reach-Avoid-Stay Specifications

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***Abstract*—Stability and safety are crucial in safety-critical control of dynamical systems. The reach-avoid-stay objectives for deterministic dynamical systems can be effectively handled by formal methods as well as Lyapunov methods with soundness and approximate completeness guarantees. However, for continuous-time stochastic dynamical systems, probabilistic reach-avoid-stay problems are viewed as challenging tasks. Motivated by the recent surge of applications in characterizing safety-critical properties using Lyapunov-barrier functions, we aim to provide a stochastic version for probabilistic reach-avoid-stay problems in consideration of robustness. To this end, based on the weak topology, we first establish a connection between probabilistic stability with safety constraints and reach-avoid-stay specifications. We, then, prove that stochastic Lyapunov-barrier functions provide sufficient conditions for the target objectives. We apply Lyapunov-barrier conditions in control synthesis for reach-avoid-stay specifications, and show its effectiveness in a case study.**

***Index Terms*—Control synthesis, probabilistic reach-avoid-stay specifications, probabilistic stability and safety with robustness, stochastic Lyapunov-barrier functions, stochastic dynamical systems, weak topology.**

I. Introduction

The reach-avoid-stay property is crucial for specifying more complex temporal logic objectives. Control synthesis of such specifications has received substantial interests in areas such as robotic motion planning [1], [2], [3]. In both deterministic and stochastic contexts, verification and control synthesis problems are achievable via abstraction-based formal methods [4], [5], [6]. Robust abstractions with soundness and approximate completeness provide guarantees for a given specification [7], [8], [9]. Despite improvements in reducing computational complexities [4], [7], [10], it still remains a fundamental challenge to overcome the curse of dimensionality in formal methods for verification and control synthesis.

On the other hand, for deterministic systems, researchers have made attempts to utilize Lyapunov-like (Lyapunov-barrier) functions to establish a connection between stability and safety attributes, and the reach-avoid-stay properties by characterizing the approximated domain [11], [12], [13]. Despite the discretization-free nature of the approach, Lyapunov and barrier functions conditions are sometimes viewed as competing objectives [11], [13]. The works [13] and [14], indicated that unifying Lyapunov and barrier functions is a nontrivial task without delving into the complicated geometry. Thanks to the fundamental converse theorems of Lyapunov and barrier functions [15], [16], [17], the theoretical work in [11] justifies that uniting Lyapunov and barrier functions is possible: A smooth Lyapunov-barrier function is sufficient and necessary (in a slightly weaker sense) for reach-avoid-stay objectives. As a step forward, Meng and Liu [18] extended the Lyapunov-barrier characterization to deterministic hybrid systems by a detailed investigation of hybrid arc topology, with validation through numerical simulations for verification and control synthesis. Meng et al. [19] applied the above theories and achieved control synthesis for reach-avoid-stay specifications in application to a system that undergoes a Hopf bifurcation. For such systems with tunable parameters, the abstraction-based algorithms underperform Lyapunov-barrier approaches due to the difficulties of adjusting the speed of the dynamical flows.

For stochastic systems, verification and control synthesis of probabilistic stability-safety type problems appear more challenging. Benchmark theorems have been established for (probabilistic) stability [20], [21] and safety verification [22], respectively, by employing stochastic Lyapunov and barrier functions. However, similar to deterministic cases, a simple combination of Lyapunov and barrier functions may not yield a sounding theoretical guarantee of stability with safety property, let alone addressing the more practical reach-avoid-stay specification. To better understand how stability and safety related notions connect in the stochastic context, this article formulates stochastic Lyapunov-barrier functions to deal with sufficient conditions for robust probabilistic reach-avoid-stay specifications. As one might expect, this task is nontrivial, which requires a comprehensive understanding of the concept of solutions, and the underlying topology that governs them.

In addition, since small perturbations should necessarily be taken into account due to reasons such as modeling uncertainties and measurement errors of the state, robust analysis provides guarantees in a worst-case scenario. Despite the current theme of regarding "inaccuracy" from the computation of probability measures as the "uncertainty" [23], [24], [25], to make a closer analogy of the deterministic case, we consider uncertainties as a result of perturbed stochastic systems, which create an inclusion of solutions. A robust satisfaction of a probabilistic specification in a perturbed stochastic system is then interpreted as follows: The solution process measured in the correspondingly worst but accurate probability law still satisfies the probabilistic specification. For a deeper comprehension of this notion of perturbation, we kindly direct readers to references [26], [27], [28], [29], and [30].

It is worth noting that the results in this article are fundamentally different from those in [20], [21], and [22] for the following reasons. 1) We consider more complicated specifications rather than merely (probabilistic) stability or safety. In particular, to utilize Lypunov-like functions for verifying probabilistic reach-avoid-stay specifications, a comprehensive analysis based on the weak topology is necessarily needed. 2) We also include small perturbations, as mentioned above, to conduct robust regularity analysis for a family of solutions rather than study a single solution at a time.

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The rest of this article is organized as follows. In Section II, we present the preliminaries for the systems, concepts of solutions, as well as other important definitions. In Section III, we show the connections between robust probabilistic reach-avoid-stay and stability with safety guarantees. In Section IV, we provide sufficient conditions for robust probabilistic reach-avoid-stay satisfactions. In Section V, a case study is conducted to demonstrate how controllers can be generated based on a control version of stochastic Lyapunov-barrier certificates. Finally, Section VI concludes this article.

Notation: We denote the Euclidean space by $\mathbb{R}^n$ for $n > 1$. We denote $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. Given $a, b \in \mathbb{R}$, we define $a \wedge b := \min(a, b)$. Let $C_b(\cdot)$ be the space of all bounded continuous functions/functionals $f : (\cdot) \to \mathbb{R}$. A continuous and strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if $\alpha(0) = 0$.

The open ball of radius r centered at x is denoted by $\mathcal{B}_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$, where $|\cdot|$ is the Euclidean norm. We also use $\mathcal{B}_r := \mathcal{B}_r(0)$ to represent open balls centered at 0. Given two sets $A, B \subseteq \mathbb{R}^n$, the set difference of B and A is defined by $B \setminus A = \{x \in B : x \notin A\}$. For a given set $A \subseteq \mathbb{R}^n$, we denote by A^c the complement of the set A (i.e., $\mathbb{R}^n \setminus A$); denote by $\bar{A}$ (respectively, ∂A) the closure (respectively, boundary) of A. For a closed set $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote the distance from x to A by $|x|_A = \inf_{y\in A} |x - y|$ and r-neighborhood of A by $\mathcal{B}_r(A) = \bigcup_{x\in A} \mathcal{B}_r(x)$.

For any stochastic processes $\{X_t\}_{t\geq 0}$, we use the shorthand notation $X := \{X_t\}_{t\geq 0}$. For any stopped process $\{X_{t\wedge\tau}\}_{t\geq 0}$, where τ is a stopping time, we use the shorthand notation X^τ. We denote the Borel σ-algebra of a set by $\mathscr{B}(\cdot)$ and the space of all probability measures on $\mathscr{B}(\cdot)$ by $\mathscr{M}(\cdot)$.

II. Preliminaries

A. System Dynamics

Consider the following perturbed stochastic differential equation (SDE):

$$dX_t = f(X_t)dt + \xi(t)dt + g(X_t)dW_t, \quad X_0 = x \tag{1}$$

where $\xi : \mathbb{R}_{\geq 0} \to \overline{\mathcal{B}}_\delta$ is any measurable point mass signal within the closed and bounded δ-ball centered at 0; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector field; $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is a smooth mapping; W is an m-dimensional Wiener process. For future references, we denote system (1) by $\mathcal{S}_\delta$, of which the $\delta \geq 0$ represents the extra δ-perturbations created by ξ.

Assumption 2.1: We make the standing assumptions on the regularity of the system $\mathcal{S}_\delta$ for the rest of this article:

i) The mappings f, g satisfy local Lipschitz continuity.
ii) The eigenvalues of the matrix $gg^T(x)$ satisfy $\sup_{x\in\mathbb{R}^n} \min_{i=1,2,\ldots,n} \lambda_i[(gg^T)(x)] > 0$.
iii) There exists a trivial solution x_e for system $\mathcal{S}_0$, such that $f(x_e) = g(x_e) = 0$.

Definition 2.2 (Characteristic operator): For each $d \in \overline{\mathcal{B}}_\delta$, we denote by $\mathcal{L}_d$ the characteristic operator of $\mathcal{S}_\delta$ as

$$\mathcal{L}_d h(x) = \nabla h(x) \cdot (f(x) + d) + \frac{1}{2} \operatorname{Tr}\left[(gg^T)(x) \cdot h_{xx}(x)\right]$$

where $h \in C^2(\mathbb{R}^n)$, $h_{xx} = (h_{x_i x_j})_{n\times n}$, and $\operatorname{Tr}[\,\cdot\,]$ denotes the trace.

Since we only care about the probabilistic properties of the state space, we consider mostly the weak solutions of the perturbed SDEs.

Definition 2.3: The system $\mathcal{S}_\delta$ admits a weak solution if there exists a (most likely unknown) filtered probability space $(\Omega^\dagger, \mathscr{F}^\dagger, \{\mathscr{F}_t^\dagger\}, \mathbb{P}^\dagger)$, where a Wiener process W is defined and a pair (X, W) is adapted, such that X solves the SDE (1) for any $\xi : \mathbb{R}_{\geq 0} \to \overline{\mathcal{B}}_\delta$.

We denote by $\Phi_\delta(x, W)$ the set of all weak solutions with $X_0 = x$ a.s. for a given $x \in \mathbb{R}^n$. Likewise, for a given set $K \subseteq \mathbb{R}^n$, let $\Phi_\delta(K, W)$ denote the set of all weak solutions with any initial distribution on $(K, \mathscr{B}(K))$.

Remark 2.4: By requiring the growth rate of nonlinear mappings and the nondegeneracy of g, i) and ii) of Assumption 2.1 ensure that, within any bounded observation region, $\mathcal{S}_\delta$ has a unique weak solution [28], [31] for any $\xi : \mathbb{R}_{\geq 0} \to \overline{\mathcal{B}}_\delta$. Specifically, condition ii) only necessitates the existence of an x where $gg^T(x)$ is positive definite.

As specified in iii), this article only considers systems with nondegenerate multiplicative noise and trivial steady point x_e for $\mathcal{S}_0$. This setup also guarantees the existence of a trivial compact steady (invariant) set containing x_e for $\mathcal{S}_\delta$. It is worth noting that additive noise, as demonstrated in the literature [32] and [33], can disrupt the stochastic stability of the steady states and, consequently, the "reach-and-stay" properties.

B. Canonical Space

We have a Wiener process W defined on some probability space $(\Omega^\dagger, \mathscr{F}^\dagger, \mathbb{P}^\dagger)$ for each weak solution. We transfer information to the canonical space, which gives us the convenience to study the law of the solution processes as well as the probabilistic behavior in the state space. Define $\Omega := C([0, \infty); \mathbb{R}^n)$ with coordinate process $\mathfrak{X}_t(\omega) := \omega(t)$ for all $t \geq 0$ and all $\omega \in \Omega$. Define $\mathcal{F}_t := \sigma\{\mathfrak{X}_s,\ 0 \leq s \leq t\}$ for each $t \geq 0$, then, the smallest σ-algebra containing the sets in every $\mathcal{F}_t$, i.e., $\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t$, turns out to be same as $\mathscr{B}(\Omega)$. For each $X \in \Phi_\delta(\mathbb{R}^n, W)$, the induced measure (law) $\mathcal{P}^X \in \mathscr{M}(\Omega)$ on $\mathcal{F}$ is such that $\mathcal{P}^X(A) = \mathbb{P}^\dagger \circ X^{-1}(A)$ for every $A \in \mathscr{B}(\Omega)$. We also denote $\mathcal{E}^X$ by the associated expectation operator w.r.t. $\mathcal{P}^X$. To emphasize the uncertainty of laws of a system $\mathcal{S}_\delta$, we prefer to work on the probability spaces $(\Omega, \mathcal{F}, \mathcal{P}^X)$ for each weak solution X rather than the original $(\Omega^\dagger, \mathscr{F}^\dagger, \mathbb{P}^\dagger)$.

Definition 2.5: (Weak convergence of measures and processes): Given any separable metric space $(\mathcal{S}, \rho)$, a sequence of $\{\mathcal{P}^n\}$ of $\mathscr{M}(\mathcal{S})$ is said to weakly converge to $\mathcal{P} \in \mathscr{M}(\mathcal{S})$, denoted by $\mathcal{P}^n \rightharpoonup \mathcal{P}$, if for all $f \in C_b(\mathcal{S})$ we have $\lim_{n\to\infty} \int_\mathcal{S} f\, d\mathcal{P}^n = \int_\mathcal{S} f\, d\mathcal{P}$. A sequence $\{X^n\}$ of continuous processes X^n with law $\mathcal{P}^n$ is said to weakly converge (on $[0, T]$) to a continuous process X with law $\mathcal{P}^X$, denoted by $X^n \rightharpoonup X$, if for all $f \in C_b(C([0, T]; \mathbb{R}^n))$ we have $\lim_{n\to\infty} \mathcal{E}^n[f(X^n)] = \mathcal{E}^X[f(X)]$.

C. Other Definitions

We first provide definitions for probabilistic set stability given a closed set $A \subseteq \mathbb{R}^n$.

Definition 2.6 (Uniform stability in probability): The set A is said to be uniformly stable in probability (Pr-U.S.) for $\mathcal{S}_\delta$ if for each $\varepsilon \in (0, 1)$ there exists $\varphi_\varepsilon \in \mathcal{K}$, such that

$$\inf_{X\in\Phi_\delta(x,W)} \mathcal{P}^X[|X_t|_A \leq \varphi_\varepsilon(|x|_A) \quad \forall t \geq 0] \geq 1 - \varepsilon \tag{2}$$

where x is the initial condition.

Remark 2.7: Equation (2) is equivalent to the following: For any $\varepsilon \in (0, 1)$ and $r > 0$, there exists an $\eta = \eta(\varepsilon, r) \in (0, r)$ such that

$$\inf_{X\in\Phi_\delta(x,W)} \mathcal{P}^X[|X_t|_A \leq r \quad \forall t \geq s(\omega)] \geq 1 - \varepsilon \tag{3}$$

whenever $|X_{s(\omega)}|_A \leq \eta$ for some random time $s(\omega)$. We can simply pick $\eta = \varphi_\varepsilon^{-1}$.

Definition 2.8 (Uniform attractivity in probability): The set A is said to be uniformly attractive in probability (Pr-U.A.) for $\mathcal{S}_\delta$ if there exists some $\eta > 0$ such that, for each $\varepsilon \in (0, 1)$, $r > 0$, there exists some

$T > 0$ such that whenever $|x|_A < \eta$,

$$\inf_{X\in\Phi_\delta(x,W)} \mathcal{P}^X[|X_t|_A < r \quad \forall t \geq T] \geq 1-\varepsilon. \tag{4}$$

Definition 2.9: (Uniformly asymptotic stability in probability): The set A is said to be uniformly asymptotically stable in probability (Pr-U.A.S.) for $\mathcal{S}_\delta$ if it is Pr-U.S. (behavior near A) and Pr-U.A. (behavior away from A) for $\mathcal{S}_\delta$.

The concept of Pr-U.A.S. extends the conventional U.A.S. as in [11], with the distinction that we measure likelihood with respect to probability measures. Next, we introduce several definitions pertinent to probabilistic stability with safety guarantees. To this end, we consider a closed unsafe set $U \subseteq \mathbb{R}^n$.

Definition 2.10 (Work place): Since the solutions are not generally nonexplosive without stability assumptions, a bounded workplace $\mathcal{R} := \mathcal{B}_{\tilde{R}}(x_e)$ with sufficiently large $\tilde{R} > 0$ is added as an extra constraint. We name $\mathcal{D} = \mathcal{D}(\mathcal{R}, U) := \mathcal{R} \cap U^c$.

Definition 2.11 (Explosion and safety): For any solution $X \in \Phi_\delta(\mathbb{R}^n, W)$, we define the corresponding explosion time $\sigma^* = \sigma^*(\mathcal{R}) := \inf\{t \geq 0 : X_t \in \mathcal{R}^c\}$ and safety time $\sigma = \sigma(\mathcal{D}) := \inf\{t \geq 0 : X_t \in \mathcal{D}^c\}$.

Remark 2.12: Safety is usually the priority in practice. Given the safety requirement w.r.t. $\mathcal{D}$ (respectively, $\mathcal{R}$), to study conditional probabilistic properties of some process X, it is equivalent to just working with the law of X^σ (respectively, X^{σ^*}). Note that for systems with trivial Pr-U.S. sets, the indicator $\mathbb{1}_{\{\sigma^*=\infty\}} \to 1$ as $R \to \infty$ and does not render "too much harm" to replace the law of X^{σ^*} by $\mathcal{P}^X$.

The following theorem verifies a notion of weak compactness of stopped weak solutions of SDE (1).

Proposition 2.13: Under the Assumption 2.1, given any compact set K, the set of all stopped process X^{σ^*} is nonempty and sequentially weakly compact (w.r.t. the weak convergence) on every filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where $X \in \bigcup_{x\in K}\Phi_\delta(x,W)$ (respectively, $X \in \Phi_\delta(K,W)$). That is, given any sequence of weak solutions $\{X^n\}_{n=1}^\infty$ in the above sense, there is a subsequence $\{X^{n_k}\}$, a process $X \in \bigcup_{x\in K}\Phi_\delta(x,W)$ (respectively, $X \in \Phi_\delta(K,W)$), such that $(X^{n_k})^{\sigma^*} \rightharpoonup X^{\sigma^*}$.

Remark 2.14: The conclusion follows immediately by [28, Theorem 1] and [29, Corollary 1.1, Ch. 3]. The proof falls in standard procedures. We can first show that the truncated laws $\{\mathcal{P}^{n,\sigma^*}\}$ of the stopped processes $\{(X^n)^{\sigma^*}\}$ form a tight family of measures on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$. Then, the relatively weak compactness follows since $(X^{n_k})^{\sigma^*} \rightharpoonup X^{\sigma^*}$ if and only if $\mathcal{P}^{n_k,\sigma^*} \rightharpoonup \mathcal{P}^{\sigma^*}$. The weak closedness comes from the compactness of the reachable sets of the stopped processes.

Now we introduce two closely related specifications pertaining to stability and safety issues.

Definition 2.15 (Probabilistic stability with safety): Given a closed set $U \subseteq \mathbb{R}^n$, let $\mathcal{D}$ and σ be defined as in Definitions 2.10 and 2.11, respectively. Given $\mathcal{X}_0, A \subseteq \mathcal{D}$ and $p \in [0,1]$, $\mathcal{S}_\delta$ is said to satisfy a probabilistic stability under safety specification w.r.t. $(\mathcal{X}_0, A, U)$ with probability at least p, denoted by $(\mathcal{X}_0, A, U, p)$, if

1) A is closed and Pr-U.A.S. for $\mathcal{S}_\delta$; and
2) For all $X \in \bigcup_{x\in\mathcal{X}_0}\Phi_\delta(x,W)$.

$$\mathcal{P}^X\left[\sigma = \infty \text{ and } \lim_{t\to\infty}|X_t|_A = 0\right] \geq p.$$

Definition 2.16: Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$. On $(\Omega, \mathcal{F})$, for each $X \in \bigcup_{x\in\mathcal{X}_0}\Phi_\delta(x,W)$, we define the events

i) $RS(\mathcal{X}_0, \Gamma, \mathcal{D}) := \{\omega : \gamma < \infty \text{ and } X_{t\wedge\sigma} \in \Gamma \forall t \geq \gamma\}$, where $\gamma := \inf\{t \geq 0 : X_t \in \Gamma\}$ of X; and
ii) $RAS(\mathcal{X}_0, \Gamma, \mathcal{D}) := RS(\mathcal{X}_0, \Gamma, \mathcal{D}) \cap \{\sigma = \infty\}$.

Definition 2.17: (Probabilistic reach-and-stay and reach-avoid-stay specification): Given a closed set $U \subseteq \mathbb{R}^n$, let $\mathcal{D}$ and σ be defined as in Definitions 2.10 and 2.11, respectively. Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$ and $p \in [0,1]$, $\mathcal{S}_\delta$ is said to satisfy a reach-avoid-stay specification w.r.t. $(\mathcal{X}_0, \Gamma, U)$ with probability at least p, denoted by $(\mathcal{X}_0, \Gamma, U, p)$, if for every $X \in \bigcup_{x\in\mathcal{X}_0}\Phi_\delta(x,W)$, we have $\mathcal{P}^X[RAS(\mathcal{X}_0, \Gamma, \mathcal{D})] \geq p$.

III. Connection to Probabilistic Stability With Safety Guarantee

A. Probabilistic Stability With Safety Implies Probabilistic Reach-Avoid-Stay

We first show that if a closed set A is Pr-U.S. for $\mathcal{S}_\delta$, then, any weak solutions starting at x from a compact subset of the p-domain of attraction is uniformly attracted to A with probability at least p.

Proposition 3.1: Suppose that a closed set $A \subset \mathcal{D}$ is Pr-U.S. for $\mathcal{S}_\delta$. Let K be a compact set and $p \in (0,1)$. Then, the following two statements are equivalent:

1) For any solution $X \in \bigcup_{x\in K}\Phi_\delta(x,W)$, we have $\mathcal{P}^X[\lim_{t\to\infty}|X_{t\wedge\sigma}|_A = 0] \geq p$.
2) For every $r > 0$, there exists $T = T(r,\varepsilon)$, such that for any $X \in \bigcup_{x\in K}\Phi_\delta(x,W)$, $\mathcal{P}^X[|X_{t\wedge\sigma}|_A < r \; \forall t \geq T] \geq p$.

Proof: Clearly (2) implies (1). We only show the converse. Suppose that (2) is not true. Then, there exists some $r > 0$ such that for all $n > 0$ there exists $x_n \in K$, $X^n \in \Phi_\delta(x_n, W)$ with law $\mathcal{P}^n$ such that

$$\mathcal{P}^n[|X^n_{t\wedge\sigma}|_A \leq r \quad \forall t \geq n] < p. \tag{5}$$

Now, let $\tau^n = \inf\{t \geq 0 : X^n_{t\wedge\sigma} \in \mathcal{B}_\eta(A)\}$, where η is to be chosen later for each n. Rearranging (5) we have

$$\begin{aligned} p &> \mathcal{P}^n[|X^n_{t\wedge\sigma}|_A \leq r \quad \forall t \geq n] \\ &\geq \mathcal{P}^n[\tau^n < n \text{ and } |X^n_{t\wedge\sigma}|_A \leq r \quad \forall t \geq n] \\ &= \mathcal{P}^n[\tau^n < n]\mathcal{P}[|X^n_{t\wedge\sigma}|_A \leq r \quad \forall t \geq n \mid \tau^n < n] \\ &\geq \mathcal{P}^n[\tau^n < n]\mathcal{P}[|X^n_{t\wedge\sigma}|_A \leq r \quad \forall t \geq \tau^n] \end{aligned} \tag{6}$$

By the definition of Pr-U.S. in view of Remark 2.7, there exists an $\eta = \eta(r,\varepsilon) < r$, such that $\mathcal{P}^n[|X^n_{t\wedge\sigma}|_A \leq r \; \forall t \geq \tau^n] \geq \varepsilon$. We choose ε sufficiently close to 1 so that, by (6), $\mathcal{P}^n[\tau^n < n] = p - \hat{p} < p$

$$\mathcal{P}^n[\tau^n < n] = p - \hat{p} < p \tag{7}$$

where $\hat{p} = \hat{p}(\varepsilon) \ll 1$. Note that we have implicitly defined η and τ^n based on the choice of ε, such that the above inequality (7) holds.

However, by Remark 2.12 and Proposition 2.13, there exists a subsequence, still denoted by $X^n \in \Phi_\delta(x_n, W)$, such that $x_n \to x$ and $(X^n)^\sigma \rightharpoonup X^\sigma$ with $X \in \Phi_\delta(x,W)$ on any compact interval of $\mathbb{R}_{\geq 0}$. By Skorohod [33, Theorem 2.4], there exists a probability space $(\tilde{\Omega}^\dagger, \tilde{\mathscr{F}}^\dagger, \{\tilde{\mathscr{F}}^\dagger_t\}, \tilde{\mathbb{P}}^\dagger)$, a process $\tilde{X}^\sigma$ and a sequence of processes $\{(\tilde{X}^n)^\sigma\}$ with laws $\mathcal{P}$ and $\{\mathcal{P}^n\}$, respectively, such that

$$\lim_{n\to\infty}(\tilde{X}^n)^\sigma = \tilde{X}^\sigma, \;\; \tilde{\mathbb{P}}^\dagger - \text{a.s.} \tag{8}$$

Let $\tau = \inf\{t \geq 0 : |\tilde{X}_{t\wedge\sigma}|_A \leq \eta/2\}$, due to the asymptotic behavior, we have $\tilde{\mathbb{P}}^\dagger[\tau < \infty] \geq p$. By this and (8), there exists some sufficiently large $N_1(\eta, q_1)$ and $N_2(\eta, q_2)$, such that for any arbitrary $q_1, q_2 \in (0,1)$

$$\tilde{\mathbb{P}}^\dagger\left[\sup_{t\in[0,N_1]}|\tilde{X}^n_{t\wedge\sigma} - \tilde{X}_{t\wedge\sigma}| \leq \eta/2\right] \geq q_1 \tag{9}$$

$$\tilde{\mathbb{P}}^\dagger[\tau < N_2] \geq pq_2. \tag{10}$$

Note that the events in (9) and (10) are independent, combining these and choosing $n \geq \max(N_1, N_2)$, we have

$$\tilde{\mathbb{P}}^{\dagger}\left[\exists t < n \text{ s.t. } \tilde{X}^n_{t\wedge\sigma} \in \mathcal{B}_\eta(A)\right] \geq \tilde{\mathbb{P}}^{\dagger}[Q] \geq pq_1q_2$$

where $Q := \{\sup_{t\in[0,N_1]} |\tilde{X}^n_{t\wedge\sigma} - \tilde{X}_{t\wedge\sigma}| \leq \eta/2\} \cap \{|\tilde{X}_{\tau\wedge\sigma}|_A \leq \eta/2$ for $\tau < N_2\}$. We let $q_1q_2 > p - \hat{p}/p$, then, there exists an n, such that $\mathcal{P}^n[\tau_n < n] > p - \hat{p}$, which contradicts (7). The proof is then completed. ■

Corollary 3.2: If $\mathcal{S}_\delta$ satisfies a stability with safety guarantee specification $(\mathcal{X}_0, A, U, p)$ and $\mathcal{X}_0$ is compact, then, for every $\varepsilon > 0$, $\mathcal{S}_\delta$ satisfies the reach-avoid-stay specification $(\mathcal{X}_0, \overline{\mathcal{B}}_\varepsilon(A), U, p)$.

Proof: Recall the safety stopping time σ in Definition 2.11. We add the condition $\{\sigma = \infty\}$, then 1) and 2) in Proposition 3.1 are still equivalent. The conclusion follows directly by the definitions of the two specifications, i.e., Definitions 2.15 and 2.17. ■

B. Converse Side

The converse side is intended to show probabilistic stability with safety is necessary for probabilistic reach-avoid-stay specifications. Unfortunately, due to the diffusion effects and the concept of weak solutions, probabilistic reach-avoid-stay specifications, other than reach-avoid-stay with probability one, may fail to be related to probabilistic stability with safety guarantees w.r.t. some subset of the target set. For this reason, we only convey the main idea due to the space limitation. The proofs are completed in [34, Appendix].

Throughout this section, we suppose that $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$ and $\mathcal{S}_\delta$ satisfies a reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, p)$. We first make a quick judgment that there exists a probability-p invariant compact subset of Γ.

Lemma 3.3: Suppose that Γ is compact and $\mathcal{X}_0$ is nonempty. If $\mathcal{S}_\delta$ satisfies a reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, p)$ with $p \in (0, 1]$, then the set

$$A = \{x \in \Gamma : \forall X \in \Phi_\delta(x, W), \mathcal{P}^X[X_t \in \Gamma \quad \forall t \geq 0] \geq p\}$$

is a nonempty and compact set with $\mathcal{P}^X[X_t \in A\ \forall t \geq 0] \geq p$ for all $X \in \Phi_\delta(A, W)$.

The next lemma shows that given an arbitrary solution X of $\mathcal{S}_{\delta'}$, we can construct a weak solution Z for $\mathcal{S}_\delta$ that solves the martingale problem and is relatively close to X.

Lemma 3.4: Let $\delta' \in (0, \delta)$ and τ be such that $\tau < \sigma$ a.s. Then, there exists some $r = r(\tau, \delta, \delta')$ such that for every $X \in \Phi_{\delta'}(x, W)$ with $x \in \mathcal{D}$ and for all $z \in \overline{\mathcal{B}}_r(x)$, there exists a weak solution $Z \in \Phi_\delta(z, W)$ such that $Z_{T\wedge\sigma} \in \overline{\mathcal{B}}_r(X_{T\wedge\sigma})$ a.s. for $T \in [\tau, \infty)$.

It can be shown that under the construction of Lemmas 3.3 and 3.4, the set A (generated by solutions of $\mathcal{S}_\delta$) is Pr-U.A. property for any weak solution of $\mathcal{S}_{\delta'}$ with $\delta' \in (0, \delta)$. However, for nonstrictly invariant sets ($p < 1$), we are not able to show the Pr-U.S. property due to a geometric gap where we cannot arbitrarily set ε and r as in Definition 2.6.

On the other hand, if there exists a invariant subset of Γ with probability one, nice properties appear. This is not a surprise given [11, Proposition 17]. The possibility of such an existence occurs when the system admits a family of a.s. stable Dirac invariant measures for each signal ξ, which are strictly contained in Γ. We convert the statement into the stochastic context in the next proposition. The proof relies on a similar construction [11, Lemma 15] as Lemma 3.4.

Proposition 3.5: For system $\mathcal{S}_\delta$, any nonempty compact set $A \subseteq \mathcal{D}$ with $\mathcal{P}^X[X_t \in A\ \forall t \geq 0] = 1$ is Pr-U.A.S. for $\mathcal{S}_{\delta'}$ whenever $\delta' \in [0, \delta)$.

Corollary 3.6: If $\mathcal{S}_\delta$ satisfies a reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, 1)$ with compact $\mathcal{X}_0$, then there exists a nonempty compact set $A \subseteq \Gamma$ with $\mathcal{P}^X[X_t \in A\ \forall t \geq 0] = 1$ such that for any $\delta' \in (0, \delta)$, $\mathcal{S}_{\delta'}$ satisfies a stability with safety specification $(\mathcal{X}_0, A, U, 1)$.

IV. Lyapunov-Barrier Characterization of Probabilistic Stability With Safety

We aim to show how Lyapunov-Barrier functions can sufficiently guarantee the probabilistic stability with safety specification in this section. To enhance clarity for readership, we begin by summarizing the frequently used notations in this section.

1) Region $\mathcal{D}$ defined in Definition 2.10 is a prescribed safety-related work place.
2) A denotes a compact set with potentially Pr-U.A.S. property.
3) V denotes a Lyapunov-like function.
4) G denotes a refined subset within $\mathcal{D}$ that will be utilized to derive the probability of stability with safety in Theorem 4.4.

Definition 4.1 (Stochastic Lyapunov functions): Let $A \subset \mathcal{D}$ be a closed set and let $R > 0$. A function $V \in C^2(\mathcal{B}_R(A); \mathbb{R}_{\geq 0})$ is said to be a stochastic Lyapunov function (SLF) w.r.t. A if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$ such that for all $x \in \mathcal{B}_R(A)$, $\alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A)$ and

$$\sup_{d\in\overline{\mathcal{B}}_\delta} \mathcal{L}_d V(x) \leq -\alpha_3(|x|_A). \tag{11}$$

We first make a quick extension of the existing Lyapunov theorems to systems with point mass perturbations.

Lemma 4.2 (Uniform recurrence): Given an SLF V, there exists some $\eta > 0$ such that, for every $\varepsilon \in (0, 1)$ and $r \in (0, R/2)$, there exists some $T = T(\varepsilon, \eta, r) > 0$ such that, for any $x \in \mathcal{B}_\eta(A)$, we have $\inf_{X\in\Phi_\delta(x,W)} \mathcal{P}^X[\tau < T] \geq 1 - \varepsilon$, where $\tau = \inf\{t \geq 0 : X_t \in \mathcal{B}_r(A)\}$ is the first hitting time of $\mathcal{B}_r(A)$ for each $X \in \Phi_\delta(x, W)$.

The proof of Lemma 4.2 is completed in [34, Appendix].

Proposition 4.3: Suppose $A \subset \mathcal{D}$ is compact. If there exists an SLF V w.r.t. A, then, A is Pr-U.A.S. for $\mathcal{S}_\delta$.

Proof: By a standard supermartingale argument [20, Lemma 1, Ch. II], we can show that the existence of SLF implies Pr-U.S. To show Pr-U.A., let $r \in (0, R/2)$, then, by Pr-U.S. and Remark 2.7, there exists a $\kappa \in (0, r)$, such that $\sup_{X\in\Phi_\delta(x,W)} \mathcal{P}^X[|X_t|_A \leq r\ \forall t \geq s(\omega)] \geq 1 - \varepsilon/2$ whenever $|X_s|_A \leq \kappa$. Now let $\tau = \inf\{t \geq 0 : X_t \in \mathcal{B}_\kappa(A)\}$. By Lemma 4.2, there exists some $\eta > 0$ such that we can find a $T = T(\varepsilon/2, \eta, \kappa)$ to make $\inf_{X\in\Phi_\delta(x,W)} \mathcal{P}^X[\tau < T] \geq 1 - \varepsilon/2$. Therefore, for all $|x|_A < \eta$ and for all $X \in \Phi_\delta(x, W)$

$$\begin{aligned}
&\mathcal{P}^X[|X_t|_A \leq r \quad \forall t \geq T] \\
&\quad \geq \mathcal{P}^X[\tau < T \text{ and } |X_t|_A \leq r \quad \forall t \geq T] \\
&\quad \geq \mathcal{P}^X[\tau < T]\mathcal{P}^X[|X_t|_A \leq r \quad \forall t \geq \tau \mid \tau < T] \\
&\quad \geq \mathcal{P}^X[\tau < T](1 - \varepsilon/2) \geq (1 - \varepsilon/2)^2 \geq 1 - \varepsilon.
\end{aligned} \tag{12}$$

■

The following result demonstrates that the existence of SLFs is sufficient to characterize probabilistic stability with safety specifications with probabilities depending on initial conditions.

Theorem 4.4: Suppose that $A \subset \mathcal{D}$ is compact. If there exists an SLF $V \in C^2(\mathcal{B}_R(A); \mathbb{R}_{\geq 0})$ and some $G := \mathcal{B}_r(A)$ such that i) $r \in (0, R]$ and $G \subset \mathcal{D}$; and ii) $\mathcal{X}_0 \subset G$, then, $\mathcal{S}_\delta$ satisfies the probabilistic stability with safety specification $(\mathcal{X}_0, A, U, 1 - \sup_{x\in\mathcal{X}_0} V(x)/\alpha_1(r))$.

We need the following lemma to accomplish the proof.

Lemma 4.5: For each $X \in \bigcup_{x\in\mathcal{X}_0} \Phi_\delta(x, W)$, set $\tau := \inf\{t \geq 0 : X_t \in G^c\}$. Then, for all $X \in \bigcup_{x\in\mathcal{X}_0} \Phi_\delta(x, W)$

$$\mathcal{P}^X\left[\lim_{t\to\infty} |X_t|_A = 0 \mid \tau = \infty\right] = 1.$$

Proof: By a similar approach to Lemma 4.2, we set arbitrary $r^* \in (0, r)$. By the Pr-U.S. property for all $X \in \Phi_\delta(x, W)$ there should exist $\eta \in (0, r^*)$ such that for any $\varepsilon \in (0, 1)$, $X_{\tau^*} \in \overline{\mathcal{B}}_\eta(A)$ implies $\mathcal{P}^X[X_t \in \mathcal{B}_{r^*}(A)\ \forall t \geq \tau^*] \geq 1 - \varepsilon$, where $\tau^* = \inf\{t \geq 0 : X_t \in \mathcal{B}_\eta(A)\}$. By Itô's formula, for each weak solution we have

$$0 \leq V(x) + \mathcal{E}^X \int_0^{\tau^* \wedge \tau \wedge t} \mathcal{L}_d V(X(s)) ds \leq V(x) - \alpha_3(\eta)\mathcal{E}^X[\tau^* \wedge \tau \wedge t]. \tag{13}$$

Since that on $\{\tau^* \wedge \tau \geq t\}$ we have $\tau^* \wedge \tau \wedge t = t$, thus

$$\mathcal{E}^X[\tau^* \wedge \tau \wedge t] \geq \int_\Omega \mathbb{1}_{\{\tau^* \wedge \tau \geq t\}} \cdot t\, d\mathcal{P}^X(\omega) = t\mathcal{P}^X[\tau^* \wedge \tau \geq t]$$

combining with (13) we have

$$\mathcal{P}^X[\tau^* \wedge \tau \geq t] \leq V(x)/t\alpha_3(\eta), \quad \text{for each } t \tag{14}$$

which implies $\mathcal{P}^X[\tau^* \wedge \tau < \infty] = 1$ for all $X \in \bigcup_{x \in \mathcal{X}_0} \Phi_\delta(x, W)$. On $\{\tau = \infty\}$, for all weak solution, we have $\mathcal{P}^X[\tau^* < \infty] = 1$ and

$$\mathcal{P}^X\left[\limsup_{t\to\infty} |X_t|_A \leq r^*\right] \geq \mathcal{P}^X[|X_t|_A \leq r^* \quad \forall t \geq \tau^* \mid \tau^* < \infty] \geq 1 - \varepsilon.$$

Since ε and r^* are arbitrary, the conclusion follows. ∎

Remark 4.6: Lemma 4.5 shows that SLFs eliminate the possibility of safe sample paths up/down-crossing any neighborhood of A infinitely often. [20, Th. 2, Ch. II] demonstrates the same result by constructing the total time spent in $G \setminus \mathcal{B}_\varepsilon(A)$ after time t and showing that it converges a.s. to 0 as $t \to \infty$.

Proof of Theorem 4.4: The existence of SLF shows that A is Pr-U.A.S. for $\mathcal{S}_\delta$. Now, for all $X \in \Phi_\delta(x, W)$ with $x \in \mathcal{X}_0$, define $\tau := \inf\{t \geq 0 : X_t \in G^c\}$. Then, for all $t \geq 0$ and for all $X \in \Phi_\delta(x, W)$

$$\mathcal{E}^X[V(X_{\tau\wedge t})] = V(X_0) + \mathcal{E}^X\left[\int_0^{\tau\wedge t} \mathcal{L}_d V(X_s) ds\right] \leq V(x)$$

and, for all $t \geq 0$

$$\mathcal{E}^X[V(X_{\tau\wedge t})] \geq \mathcal{E}^X[\mathbb{1}_{\{\tau \leq t\}} V(X_\tau)] > \alpha_1(r)\mathcal{P}^X[\tau \leq t]$$

which imply $\mathcal{P}^X[\tau \leq t] < V(x)/\alpha_1(r)\ \ \forall t \geq 0$. Sending $t \to \infty$ we get for all $X \in \bigcup_{x\in\mathcal{X}_0} \Phi_\delta(x, W)$ and $\mathcal{P}^X[\tau < \infty] < V(x)/\alpha_1(r)$, i.e.,

$$\inf_{X \in \Phi_\delta(x,W)} \mathcal{P}^X[\tau = \infty] \geq 1 - \frac{\sup_{x\in\mathcal{X}_0} V(x)}{\alpha_1(r)}.$$

Since $\{\tau = \infty\} \subset \{\sigma = \infty\}$ and by Lemma 4.5, the conclusion follows. ∎

From an application perspective, the condition of the type (11) can be stringent for large δ, potentially leading to the nonexistence of a Lyapunov-type function V. However, this section provides a perspective that one can directly work on a family of uncertain processes, which solves (1) that is also affected by extra perturbations, and analyze their probabilistic regularities comprehensively. In particular, (11) is the key to the regularity analysis in view of the concise result in Lemma 4.5. These theoretical guarantees regarding probabilistic reach-avoid-stay specifications, given the existence of V, possesses greater practical value in verification and control synthesis when considering small given small δ (including the case when $\delta = 0$). On the other hand, if we were to relax (11) to $\mathcal{L}V(x) \leq -\alpha_3(|x|_A)$, the robustness analysis could be more tedious given that Lemma 4.5 may not be guaranteed.

In the proof, we have also seen that conditions $\alpha_1(x) \leq V(x)$ and $\sup_{d\in\overline{\mathcal{B}}_\delta} \mathcal{L}_d V \leq 0$ play the role of guaranteeing the probabilistic set invariance. We refer to these conditions as the stochastic barrier certificates. An application in control synthesis, termed stochastic control barrier functions, has been shown in [35, Proposition III.8] with better safety probability compared with the zeroing-type barrier certificates [36], however, less effective than the reciprocal-type barrier certificates (see the definition in [35, Definition III.1]). A thorough comparison between the abovementioned stochastic barrier functions can be found in [35]. To provide stability with safety with probability 1, one can combine SLF with the reciprocal-type barrier functions.

Theorem 4.7: Under the same assumption in Theorem 4.4. Suppose there exists an SLF $V \in C^2(\mathcal{B}_R(A); \mathbb{R}_{\geq 0})$, some $G := \mathcal{B}_r(A)$ such that $G \in \mathcal{D}$ and $\mathcal{X}_0 \subset G$, as well as a function $B \in C^2(G; \mathbb{R}_{\geq 0})$ satisfying

i) $\exists \alpha_1, \alpha_2 \in \mathcal{K}$ s.t.

$$\frac{1}{\alpha_1(|x|_A)} \leq B(x) \leq \frac{1}{\alpha_2(|x|_A)} \quad \forall x \in G \tag{15}$$

ii) $\exists \alpha_3 \in \mathcal{K}$ s.t.

$$\sup_{d\in\overline{\mathcal{B}}_\delta} [\mathcal{L}_d B(x) - \alpha_3(|x|_A)] \leq 0 \quad \forall x \in G. \tag{16}$$

Then, $\mathcal{S}_\delta$ satisfies the probabilistic stability with safety specification $(\mathcal{X}_0, A, U, 1)$.

Proof: The proof is similar to Theorem 4.4. We rely on the SLF to provide the property shown in Lemma 4.5. Then, the reciprocal type barrier function B guarantees that $\mathcal{P}[\tau = \infty] = 1$ [22, Th. 1] for each weak solution. ∎

Remark 4.8: Suppose $U^c := \{x \in \mathbb{R}^n : h(x) \geq 0\}$ where h is smooth, one can possibly enlarge G, such that $G \cap U \neq \emptyset$ with $\partial(G \cap U)$ being piecewise smooth. To see the satisfaction of stability with safety specifications, along with the old conditions, one can introduce an extra reciprocal-type barrier function, denoted by $\tilde{B}$, and verify extra conditions that are similar to (15) and (16) by replacing $|x|_A$ with $h(x)$.

V. Applications in Control Problems

In this section, based on the results from Sections III and IV, we make a straightforward extension to a stochastic control Lyapunov-barrier characterization for $\mathcal{S}_\delta$ satisfying a probabilistic reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, p)$ under controls. As a continuation of [19], we conduct a case study on enhancing the performance of jet engine compressors under both noisy disturbances and bounded point mass perturbations, based on a reduced Moore–Greitzer nonlinear SDE model.

A. *Probabilistic Reach-Avoid-Stay Control via Stochastic Control Lyapunov-Barrier Functions*

We first recast the notion from Section II for control systems. Given a nonempty compact convex set of control inputs $\mathscr{U} \subset \mathbb{R}^p$, consider a nonlinear system of the form

$$dX_t = f(X_t)dt + \xi(t)dt + b(X_t)udt + g(X_t)dW_t \tag{17}$$

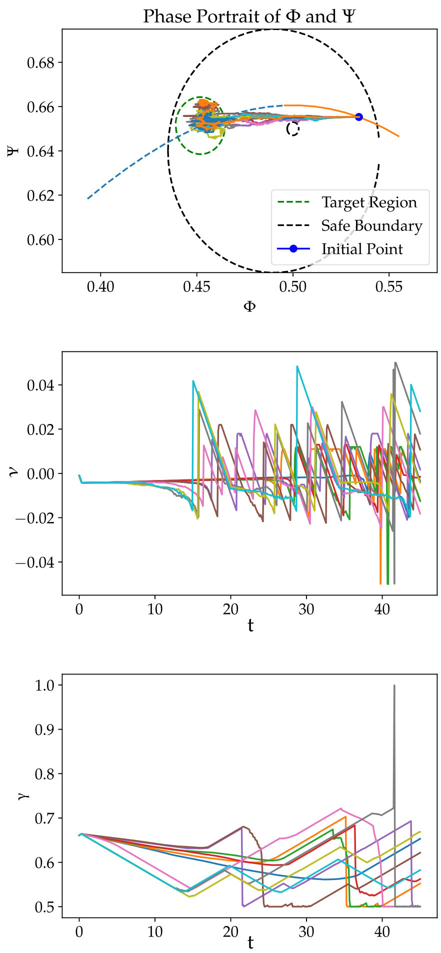

Fig. 1. Cluster of controlled sample paths, control inputs v and γ of Problem 5.6.

where $X_0 = x$; the mapping $b : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is smooth; $u : \mathbb{R}_{\geq 0} \to \mathscr{U}$ is a locally bounded measurable control signal, whilst the other notation remains the same.

Definition 5.1 (Control strategy): A control strategy is a function

$$\kappa : \mathbb{R}^n \to \mathscr{U}. \tag{18}$$

We further denote $\mathcal{S}_\delta^\kappa$ by the control system driven by (17) that is comprised by $u = \kappa(x)$.

Definition 5.2. (Probabilistic reach-avoid-stay controllable): Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$, and $p \in [0, 1]$, $\mathcal{S}_\delta$ is said to be probabilistic reach-avoid-stay controllable w.r.t. $(\mathcal{X}_0, \Gamma, U, p)$, if there exists a Lipschitz continuous control strategy κ, such that the system $\mathcal{S}_\delta^\kappa$ satisfies the specification $(\mathcal{X}_0, \Gamma, U, p)$.

We have the following theoretical guarantee regarding the aforementioned probabilistic reach-avoid-stay problem by employing the Lyapunov method.

Proposition 5.3: Let $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$ be compact, where $\Gamma \supsetneq A$ for some compact set $A \subset \mathcal{D}$. If there exists a smooth function $V \in C^2(\mathcal{B}_R(A); \mathbb{R}_{\geq 0})$ and some $G := \mathcal{B}_r(A) \subset S$, such that

i) $r \in (0, R]$, $G \subset \mathcal{D}$, and $\mathcal{X}_0 \subset G$;

ii) $\alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A)$ and

$$\inf_{u\in\mathscr{U}} \sup_{x\in S} \sup_{d\in\delta\mathcal{B}} [\mathcal{L}_d^u V(x) + \alpha_3(|x|_A)] \leq 0$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$, where $\mathcal{L}_d^u V(x) := \mathcal{L}_d V(x) + \nabla V(x) \cdot b(x)u$.

Then, $\mathcal{S}_\delta$ is probabilistically reach-avoid-stay controllable w.r.t. $(\mathcal{X}_0, \Gamma, U, 1 - \sup_{x\in\mathcal{X}_0} V(x)/\alpha_1(r))$.

Proof: The proof is a direct application of Corollary 3.2 and Theorem 4.4. Note that Theorem 4.4 only establishes that the existence of V ensures probabilistic stability with a safety property. In order to utilize Corollary 3.2, which establishes the connection between the two specifications, it is necessary for the set Γ to be compact and strictly contain A. ■

Remark 5.4: Similarly, one can extend the above proposition to find sufficient conditions for a "Probability 1" reach-avoid-stay based on Theorem 4.7. Apart from the conditions in Proposition 5.3, one need to additionally verify if there exists a $B \in C^2(G; \mathbb{R}_{\geq 0})$ satisfying

i) $1/\tilde{\alpha}_1(|x|_A) \leq B(x) \leq 1/\tilde{\alpha}_2(|x|_A)$ $\forall x \in G$ for some class-$\mathcal{K}$ functions $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$;

ii) $\inf_{u\in\mathscr{U}} \sup_{x\in S} \sup_{d\in\delta\mathcal{B}} [\mathcal{L}_d^u B(x) - \tilde{\alpha}_3(|x|_A)] \leq 0$, for some class-$\mathcal{K}$ function $\tilde{\alpha}_3$.

B. Case Study

We use the reduced Moore–Greitzer SDE model with an additive control input $[v, 0]^T$ and a multiplicative noise to illustrate the effectiveness. The model is given as

$$\frac{d}{dt}\begin{bmatrix}\Phi(t)\\ \Psi(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{l_c}(\psi_c - \Psi(t))\\ \frac{1}{16 l_c}\left(\Phi(t) - \gamma\sqrt{\Psi(t)}\right)\end{bmatrix} + \begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix} + \varepsilon\begin{bmatrix}(\Phi(t) - \Phi_e(\gamma))\beta_1(t)\\ (\Psi(t) - \Psi_e(\gamma))\beta_2(t)\end{bmatrix} + \begin{bmatrix}v(t)\\ 0\end{bmatrix} \tag{19}$$

where $\psi_c = a + \iota * [1 + 3/2(\Phi/\Theta - 1) - 1/2(\Phi/\Theta - 1)^3]$, β_1 and β_2 are i.i.d. Brownian motions, $(\Phi_e(\gamma), \Psi_e(\gamma)) =: X_e(\gamma)$ are equilibrium points for $\xi_1, \xi_2, v \equiv 0$. The other parameters are as follows:

$$l_c = 8,\ \iota = 0.18,\ \Theta = 0.25,\ a = 0.67\iota,\ \varepsilon = 0.08,\ \text{and}\ \delta = 0.01.$$

The physical meanings of variables, parameters and the description of this model can be found in [19, Section V]. The full Moore–Greitzer model, as described in [37] and [38], includes a coupled infinite-dimensional state. Parameters are chosen based on the criteria introduced in [38] to ensure that the infinite-dimensional state quickly converges to zero and decouples from the 2-D sub-system (19).

Remark 5.5: For $\xi_1, \xi_2, v \equiv 0$, the system admits a family of equilibrium points $X_e(\gamma)$ depending on the tunable parameter γ. As γ drops in the neighborhood of the deterministic Hopf bifurcation point, the system undergoes a D-bifurcation (the stability of the invariant measure $\delta_{\{X_e\}}$ changes and a new invariant measure in $\mathbb{R}^n \setminus \{X_e\}$ is built up) and a P-bifurcation (the shape of the density of the new measure changes). The full stochastic Hopf bifurcation diagram in [39, Fig 9.13] conveys the brief idea.

Within the a.s. exponentially stable region, any bounded perturbation ξ causes a bounded long-term perturbation of $X_e(\gamma)$, and ultimately formulates a compact set containing $X_e(\gamma)$. For unstable $\delta_{\{X_e\}}$, especially for those after P-bifurcation (which is numerically verified to be larger than $\gamma = 0.609$ as demonstrated in [40, Fig. 3.2]), we are interested in stabilizing the robust system to a compact set.

Problem 5.6: We aim to manipulate γ and v simultaneously, such that the state (Φ, Ψ) are regulated to satisfy reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, 1)$. We require that $\gamma : \mathbb{R}_{\geq 0} \to [0.5, 1]$ is time-varied with $\gamma(0) \in [0.62, 0.66]$ and $|\gamma(t + \tau) - \gamma(t)| \leq 0.01\tau$ for any $\tau > 0$. We define $\mathcal{X}_0 = \{(\Phi_e(\gamma(0)), \Psi_e(\gamma(0)))\}$; Γ to be the ball that centered at $\varrho = (0.4519, 0.6513)$ with radius $r = 0.013$, i.e., $\Gamma = \varrho + r\overline{\mathcal{B}}$; the unsafe set $U = \{(x, y) : h_1 \leq 0\} \cap \{(x, y) : h_2 \leq 0\}$, where $h_1(x,y) = -|(x,y) - (0.49, 0.64)| + 0.055$ and $h_2(x,y) = |(x,y) - (0.50, 0.65)| - 0.003$. We set $v \in \mathscr{U} = [-0.05, 0.05]$.

We refer readers to [19, Remark 30] for treatments of μ as another control input. For each SDE, the signals ξ_1 and ξ_2 of each sampling time is generated randomly from $\{-0.1, 0.1\}$.

In view of Proposition 5.3 and Remark 5.4, we choose SLF $V(x,y) = l_c/2(x-\varrho_1)^2 + 8l_c(y-\varrho_2)^2$ and $\alpha_3(x) = 0.1x$; set $B_i = -\log(h_i/1 + h_i)$ for $i = 1, 2$. Note that, with multiplicative noise in (19), the validity of the quadratic function V can be verified using similar procedures as demonstrated in various examples in [21, Chapter 4]. The validity of the choice of B_i, for $i = 1, 2$, is also established in [22, Sec. VII]. Due to space limitations, we have omitted the calculus involved but kindly refer readers to [21] and [22] for a detailed verification. The settings for the quadratic programming remain the same as [19, Section V.B]. We mix sample paths under different ξ_1 and ξ_2 and show the simulation results in Fig. 1. It can be demonstrated that, under valid control inputs, all controlled sample paths satisfy the specified probabilistic reach-avoid-stay property outlined in Problem 5.6.

Remark 5.7: Note that we have adopted reciprocal-type barrier functions, which potentially generate impulse-like control signals (to cancel the diffusion effects) and terminate the programming. However, once the synthesis succeeds, the feasible controlled sample paths satisfy the specification.

VI. Conclusion

In this article, we formulated stochastic Lyapunov-barrier functions to develop sufficient conditions for probabilistic reach-avoid-stay specifications. Given the uncertainties of the model, robustness was taken into account such that a worst-case scenario is guaranteed. We characterized a general topological structure of the initial sets, target sets, and unsafe sets under the stochastic settings and discussed relaxations given the smoothness of the unsafe boundary. We investigated the effectiveness in a case study of jet engine compressor control problem. Despite the potentially unbounded control inputs, the control version of SLF, along with reciprocal-type barrier functions, guarantees a Probability-1 satisfaction.

However, like deterministic Lyapunov-like functions only providing a stability characterization of the solutions, the stochastic Lyapunov-type argument can only estimate a lower bound of "satisfaction in probability" without solving the evolving states and distributions. It renders difficulties of selecting Lyapunov/barrier functions under the restrictive geometric requirements of the initial conditions and unsafe sets.

For future work, compared with the rough estimation of "probabilistic domain of satisfactions" given Lyapunov-like functions, it would be necessary to consider accurate evaluation of laws and investigate formal methods in providing more reliable schemes for finding the probabilistic winning sets (from which the specifications are satisfied). Algorithms should be developed based on sound and (approximately) complete stochastic abstractions for robust stochastic control synthesis problems.

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