

Sufficient Conditions for Robust Probabilistic Reach-Avoid-Stay Specifications using Stochastic Lyapunov-Barrier Functions

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Abstract—Stability and safety are crucial in safety-critical control of dynamical systems. The reach-avoid-stay objectives for deterministic dynamical systems can be effectively handled by formal methods as well as Lyapunov methods with soundness and approximate completeness guarantees. However, for continuous-time stochastic dynamical systems, probabilistic reach-avoid-stay problems are viewed as challenging tasks. Motivated by the recent surge of applications in characterizing safety-critical properties using Lyapunov-barrier functions, we aim to provide a stochastic version for probabilistic reach-avoid-stay problems in consideration of robustness. To this end, we first establish a connection between probabilistic stability with safety constraints and reach-avoid-stay specifications. We then prove that stochastic Lyapunov-barrier functions provide sufficient conditions for the target objectives. We apply Lyapunov-barrier conditions in control synthesis for reach-avoid-stay specifications, and show its effectiveness in a case study.

Index Terms—Stochastic dynamical systems; probabilistic reach-avoid-stay specifications; probabilistic stability and safety with robustness; stochastic Lyapunov-barrier functions.

I. INTRODUCTION

The reach-avoid-stay property is one of the building blocks for specifying more complex temporal logic objectives. Control synthesis of such specifications has received substantial interests in areas such as robotic motion planning [8], [7], [24].

In the deterministic context, verification and control synthesis problems are achievable via abstraction-based formal methods [3]. Considering uncertain transition systems with bounded measurable signals, robust abstractions with soundness and approximate completeness provide guarantees for a given specification [17], [15]. Despite algorithmic improvements on reducing computational complexities [9], [17], [3], it still remains a fundamental challenge to overcome the curse of dimensionality in abstraction-based approaches for verification and control synthesis.

On the other hand, Lyapunov-like functions are able to connect stability and safety attributes with reach-avoid-stay properties in terms of characterizing the approximated domain of attraction [19], [27]. Thanks to the fundamental converse theorems of Lyapunov and barrier functions [29], [16], [18], the theoretical work in [19] justifies that a smooth Lyapunov-barrier function is sufficient and necessary (in a slightly weaker sense) for reach-avoid-stay objectives. In terms of programming, the framework [1] also shows effectiveness in the control synthesis of safety and stabilization

objectives without spatial discretization. The recent work [22] took advantages of the above and achieved control synthesis for reach-avoid-stay specifications in application to a system that undergoes a Hopf-bifurcation. For such systems with tunable parameters, the abstraction-based algorithms underperform Lyapunov-barrier approaches due to the difficulties of adjusting the speed of the dynamical flows.

As for verification and control synthesis of probabilistic stability-safety type problems, it appears more challenging. Authors in [14], [4], [20], [5] applied abstract models, such as interval-valued Markov chain and bounded-parameter Markov decision process, on discrete-time continuous-state stochastic systems to compute an inclusion of the real satisfying probability and synthesize controllers for probabilistic specifications (including probabilistic reachability on an infinite horizon). Works in [26], [30], [10] characterized value functions for reachability/reach-avoid problems in discrete-time continuous-state stochastic systems and applied dynamic programming for synthesizing optimal controllers.

Since small perturbations should necessarily be taken into account due to reasons such as modelling uncertainties and measurement errors of the state, robust analysis provides guarantees in a worst-case scenario. Despite the current theme of regarding ‘inaccuracy’ from the computation of probability measures as the ‘uncertainty’ [14], [4], [20], [5], to make a closer analogy of the deterministic case, we consider uncertainties as a result of perturbed stochastic systems which create an inclusion of solutions. A robust satisfaction of a probabilistic specification in a perturbed stochastic system is then interpreted as follows: the solution process measured in the correspondingly worst but accurate probability law still satisfies the probabilistic specification. The work [28] demonstrated the robust Lyapunov-stability for discrete-time stochastic systems with perturbations. Continuous-time stochastic differential inclusions are also well studied [11], [12], [21].

Motivated by the deterministic robust abstractions [17], [15] and the recent comparisons with robust Lyapunov-type characterizations of reach-avoid-stay specifications [22], to better understand how these two connect in the stochastic context, this paper formulates stochastic Lyapunov-barrier functions to deal with sufficient conditions for robust probabilistic reach-avoid-stay specifications.

The rest of this paper is organized as follows. In Section II, we present the preliminaries for the systems, concepts of solutions, as well as other important definitions. In Section III, we show the connections between robust probabilistic reach-avoid-stay and stability with safety guarantees. In Section

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IV, we provide sufficient conditions for robust probabilistic reach-avoid-stay satisfactions. In section V, a case study on a stochastic Moore-Greitzer model is conducted to demonstrate how controllers can be generated based on a control version of stochastic Lyapunov-barrier certificates. The paper is concluded in Section VI. Due to space limitation, proofs are omitted and can be found in the arXiv version [23].

Notation: We denote the Euclidean space by \mathbb{R}^n for $n > 1$. We denote \mathbb{R} the set of real numbers, and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. Given $a, b \in \mathbb{R}$, we define $a \wedge b := \min(a, b)$. Let $C_b(\cdot)$ be the space of all bounded continuous functions/functionals $f : (\cdot) \rightarrow \mathbb{R}$. A continuous and strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to class \mathcal{K} if $\alpha(0) = 0$.

The open ball of radius r centered at x is denoted by $\mathcal{B}_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$, where $|\cdot|$ is the Euclidean norm. We also use $\mathcal{B}_r := \mathcal{B}_r(0)$ to represent open balls centered at 0. Given two sets $A, B \subseteq \mathbb{R}^n$, the set difference of B and A is defined by $B \setminus A = \{x \in B : x \notin A\}$. For a given set $A \subseteq \mathbb{R}^n$, we denote by A^c the complement of the set A (i.e., $\mathbb{R}^n \setminus A$); denote by \bar{A} (resp. ∂A) the closure (resp. boundary) of A . For a closed set $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote the distance from x to A by $|x|_A = \inf_{y \in A} |x - y|$ and r -neighborhood of A by $\mathcal{B}_r(A) = \bigcup_{x \in A} \mathcal{B}_r(x)$.

For any stochastic processes $\{X_t\}_{t \geq 0}$ we use the shorthand notation $X := \{X_t\}_{t \geq 0}$. For any stopped process $\{X_{t \wedge \tau}\}_{t \geq 0}$, where τ is a stopping time, we use the shorthand notation X^τ . We denote the Borel σ -algebra of a set by $\mathcal{B}(\cdot)$ and the space of all probability measures on $\mathcal{B}(\cdot)$ by $\mathcal{M}(\cdot)$.

II. PRELIMINARIES

A. System dynamics

Consider the following perturbed stochastic differential equation (SDE):

$$dX_t = f(X_t)dt + \xi(t)dt + g(X_t)dW_t, \quad X_0 = x, \quad (1)$$

where $\xi : \mathbb{R}_{\geq 0} \rightarrow \bar{\mathcal{B}}_\delta$ is any measurable point mass signal; $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector field; $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ is a smooth mapping; W is an m -dimensional Wiener process. For future references, we denote systems driven by SDE (1) by \mathcal{S}_δ , of which the δ represents the δ -perturbations.

Assumption 2.1: We make the standing assumptions on the regularity of the system \mathcal{S}_δ for the rest of this paper:

- (i) The mappings f, g satisfy local Lipschitz continuity.
- (ii) The eigenvalues $\lambda_i[(gg^T)(x)]$ of the matrix $gg^T(x)$ for $i = 1, 2, \dots, n$ satisfy

$$\sup_{x \in \mathbb{R}^n} \min_{i=1,2,\dots,n} \lambda_i[(gg^T)(x)] > 0.$$

- (iii) There exists a trivial solution x_e for system \mathcal{S}_0 such that $f(x_e) = g(x_e) = 0$.

Definition 2.2 (Infinitesimal generator): For each $d \in \bar{\mathcal{B}}_\delta$, we denote by \mathcal{L}_d the infinitesimal generator of \mathcal{S}_δ as

$$\mathcal{L}_d h(x) = \nabla h(x) \cdot (f(x) + d) + \frac{1}{2} \text{Tr}[(gg^T)(x) \cdot h_{xx}(x)],$$

where $h \in C^2(\mathbb{R}^n)$, $h_{xx} = (h_{x_i x_j})_{n \times n}$, and $\text{Tr}[\cdot]$ denotes the trace.

Since we only care about the probabilistic properties in the state space, we consider mostly the weak solutions of the perturbed SDEs.

Definition 2.3: The system \mathcal{S}_δ admits a weak solution if there exists a (most likely unknown) filtered probability space $(\Omega^\dagger, \mathcal{F}^\dagger, \{\mathcal{F}_t^\dagger\}, \mathbb{P}^\dagger)$, where a Wiener process W is defined and a pair (X, W) is adapted, such that X solves the SDE (1) for any $\xi : \mathbb{R}_{\geq 0} \rightarrow \bar{\mathcal{B}}_\delta$.

We denote by $\Phi_\delta(x, W)$ the set of all weak solutions with $X_0 = x$ a.s. for a given $x \in \mathbb{R}^n$. Likewise, for a given set $K \subseteq \mathbb{R}^n$, let $\Phi_\delta(K, W)$ denote the set of all weak solutions with any initial distribution on $(K, \mathcal{B}(K))$.

B. Canonical space

We have a Wiener process W defined on some probability space $(\Omega^\dagger, \mathcal{F}^\dagger, \mathbb{P}^\dagger)$ for each weak solution. We transfer information to the canonical space, which gives us the convenience to study the law of the solution processes as well as the probabilistic behavior in the state space. Define $\Omega := C([0, \infty); \mathbb{R}^n)$ with coordinate process $\mathfrak{X}_t(\omega) := \omega(t)$ for all $t \geq 0$ and all $\omega \in \Omega$. Define $\mathcal{F}_t := \sigma\{\mathfrak{X}_s, 0 \leq s \leq t\}$ for each $t \geq 0$, then the smallest σ -algebra containing the sets in every \mathcal{F}_t , i.e. $\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$, turns out to be same as $\mathcal{B}(\Omega)$. For each $X \in \Phi_\delta(\mathbb{R}^n, W)$, the induced measure (law) $\mathcal{P}^X \in \mathcal{M}(\Omega)$ on \mathcal{F} is such that $\mathcal{P}^X(A) = \mathbb{P}^\dagger \circ X^{-1}(A)$ for every $A \in \mathcal{B}(\Omega)$. We also denote \mathcal{E}^X by the associated expectation operator w.r.t. \mathcal{P}^X .

To emphasize on the uncertainty of laws of a system \mathcal{S}_δ , we prefer to work on the probability spaces $(\Omega, \mathcal{F}, \mathcal{P}^X)$ for each weak solution X rather than the original $(\Omega^\dagger, \mathcal{F}^\dagger, \mathbb{P}^\dagger)$.

Definition 2.4: (Weak convergence of measures and processes): Given any separable metric space (\mathcal{S}, ρ) , a sequence of $\{\mathcal{P}^n\}$ of $\mathcal{M}(\mathcal{S})$ is said to weakly converge to $\mathcal{P} \in \mathcal{M}(\mathcal{S})$, denoted by $\mathcal{P}^n \rightharpoonup \mathcal{P}$, if for all $f \in C_b(\mathcal{S})$ we have $\lim_{n \rightarrow \infty} \int_{\mathcal{S}} f d\mathcal{P}^n = \int_{\mathcal{S}} f d\mathcal{P}$. A sequence $\{X^n\}$ of continuous processes X^n with law \mathcal{P}^n is said to weakly converge (on $[0, T]$) to a continuous process X with law \mathcal{P}^X , denoted by $X^n \rightharpoonup X$, if for all $f \in C_b(C([0, T]; \mathbb{R}^n))$ we have $\lim_{n \rightarrow \infty} \mathcal{E}^n[f(X^n)] = \mathcal{E}^X[f(X)]$.

C. Strong Markov properties

Since for each measurable signal ξ , the corresponding martingale problem is well posed under Assumption 2.1, by Markovian selection theorems [6, Theorem 5.19, Chap 4], the unique solution \mathcal{P} to the martingale problem also makes the weak solution (X, W) Markovian.

D. Other definitions

We first provide definitions for probabilistic set stability given a closed set $A \subseteq \mathbb{R}^n$.

Definition 2.5 (Uniform stability in law): The set A is said to be uniformly stable in law (Pr-U.S.) for \mathcal{S}_δ if for each $\varepsilon \in (0, 1)$ there exists $\varphi_\varepsilon \in \mathcal{K}$ such that

$$\inf_{X \in \Phi_\delta(x, W)} \mathcal{P}^X[|X_t|_A \leq \varphi_\varepsilon(|x|_A) \forall t \geq 0] \geq 1 - \varepsilon, \quad (2)$$

where x is the initial condition.

Remark 2.6: Equation (2) is equivalent to the following: for any $\varepsilon \in (0, 1)$ and $r > 0$, there exists an $\eta = \eta(\varepsilon, r) \in (0, r)$ such that

$$\inf_{X \in \Phi_\delta(x, W)} \mathcal{P}^X [|X_t|_A \leq r \ \forall t \geq s(\omega)] \geq 1 - \varepsilon, \quad (3)$$

whenever $|X_{s(\omega)}|_A \leq \eta$. We can simply pick $\eta = \varphi_\varepsilon^{-1}$.

Definition 2.7 (Uniform attractivity in law): The set A is said to be uniformly attractive in law (Pr-U.A.) for \mathcal{S}_δ if there exists some $\eta > 0$ such that, for each $\varepsilon \in (0, 1)$, $r > 0$, there exists some $T > 0$ such that whenever $|x|_A < \eta$,

$$\inf_{X \in \Phi_\delta(x, W)} \mathcal{P}^X [|X_t|_A < r, \ \forall t \geq T] \geq 1 - \varepsilon. \quad (4)$$

Definition 2.8 (Uniformly asymptotic stability in law):

The set A is said to be uniformly asymptotically stable in law (Pr-U.A.S.) for \mathcal{S}_δ if it is Pr-U.S. and Pr-U.A. for \mathcal{S}_δ .

Next we introduce several definitions pertinent to probabilistic stability with safety guarantees. To this end, we consider a closed unsafe set $U \subseteq \mathbb{R}^n$.

Definition 2.9 (Work place): Since the solutions are not generally non-explosive without stability assumptions, a bounded workplace $\mathcal{R} := \mathcal{B}_{\bar{R}}(x_e)$ with sufficiently large $\bar{R} > 0$ is added as an extra constraint. We name $\mathcal{D} = \mathcal{D}(\mathcal{R}, U) := \mathcal{R} \cap U^c$.

Definition 2.10 (Explosion and safety): For any solution $X \in \Phi_\delta(\mathbb{R}^n, W)$, we define the corresponding explosion time $\sigma^* = \sigma^*(\mathcal{R}) := \inf\{t \geq 0 : X_t \in \mathcal{R}^c\}$ and safety time $\sigma = \sigma(\mathcal{D}) := \inf\{t \geq 0 : X_t \in \mathcal{D}^c\}$.

Remark 2.11: Safety is usually the priority in practice. Given safety requirement w.r.t. \mathcal{D} (resp. \mathcal{R}), to study conditional probabilistic properties of some process X , it is equivalent to just working with the law of X^σ (resp. X^{σ^*}). Note that for systems with trivial Pr-U.S. sets, the indicator $\mathbb{1}_{\{\sigma^* = \infty\}} \rightarrow 1$ as $R \rightarrow \infty$ and does not render ‘too much harm’ to replace the law of X^{σ^*} by \mathcal{P}^X .

The following theorem verifies a notion of weak compactness of stopped weak solutions of SDE (1).

Proposition 2.12: Under the Assumption 2.1, given any compact set K , the subset of all stopped process X^{σ^*} is nonempty and sequentially weakly compact w.r.t. the convergence in law on every filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]})$, where $X \in \bigcup_{x \in K} \Phi_\delta(x, W)$ (resp. $X \in \Phi_\delta(K, W)$). That is, given any sequence of weak solutions $\{X^n\}_{n=1}^\infty$ in the above sense, there is a subsequence $\{X^{n_k}\}$, a process $X \in \bigcup_{x \in K} \Phi_\delta(x, W)$ (resp. $X \in \Phi_\delta(K, W)$) such that $(X^{n_k})^{\sigma^*} \rightharpoonup X^{\sigma^*}$.

Remark 2.13: The conclusion follows immediately by [11, Theorem 1] and [12, Corollary 1.1, Chap 3]. The proof falls in standard procedures. We can first show that the truncated laws $\{\mathcal{P}^{n, \sigma^*}\}$ of the stopped processes $\{(X^n)^{\sigma^*}\}$ form a tight family of measures on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]})$. Then the relatively weak compactness follows since $(X^{n_k})^{\sigma^*} \rightharpoonup X^{\sigma^*}$ if and only if $\mathcal{P}^{n_k, \sigma^*} \rightharpoonup \mathcal{P}^{\sigma^*}$. The weak closedness comes from compactness of the reachable sets of the stopped processes.

Definition 2.14 (Probabilistic stability with safety):

Given a closed set $U \subseteq \mathbb{R}^n$, let \mathcal{D} and σ be defined as in

Definitions 2.9 and 2.10, respectively. Given $\mathcal{X}_0, A \subseteq \mathcal{D}$ and $p \in [0, 1]$, \mathcal{S}_δ is said to satisfy a probabilistic stability under safety specification w.r.t. (\mathcal{X}_0, A, U) with probability at least p , denoted by (\mathcal{X}_0, A, U, p) , if

- (1) A is closed and Pr-U.A.S. for \mathcal{S}_δ ;
- (2) For all $X \in \bigcup_{x \in \mathcal{X}_0} \Phi_\delta(x, W)$,

$$\mathcal{P}^X \left[\sigma = \infty \text{ and } \lim_{t \rightarrow \infty} |X_t|_A = 0 \right] \geq p.$$

Definition 2.15: Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$. On (Ω, \mathcal{F}) , for each $X \in \bigcup_{x \in \mathcal{X}_0} \Phi_\delta(x, W)$, we define the events

- (i) $RS(\mathcal{X}_0, \Gamma, \mathcal{D}) := \{\omega : \gamma < \infty \text{ and } X_{t \wedge \sigma} \in \Gamma \ \forall t \geq \gamma\}$, where $\gamma := \inf\{t \geq 0 : X_t \in \Gamma\}$ of X ;
- (ii) $RAS(\mathcal{X}_0, \Gamma, \mathcal{D}) := RS(\mathcal{X}_0, \Gamma, \mathcal{D}) \cap \{\sigma = \infty\}$.

Definition 2.16: (Probabilistic reach-and-stay and reach-avoid-stay specification): Given a closed set $U \subseteq \mathbb{R}^n$, let \mathcal{D} and σ be defined as in Definitions 2.9 and 2.10, respectively. Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$ and $p \in [0, 1]$, \mathcal{S}_δ is said to satisfy a reach-avoid-stay specification w.r.t. $(\mathcal{X}_0, \Gamma, U)$ with probability at least p , denoted by $(\mathcal{X}_0, \Gamma, U, p)$, if for every $X \in \bigcup_{x \in \mathcal{X}_0} \Phi_\delta(x, W)$, we have $\mathcal{P}^X [RAS(\mathcal{X}_0, \Gamma, \mathcal{D})] \geq p$.

III. A CONNECTION TO PROBABILISTIC STABILITY WITH SAFETY GUARANTEE

We intend to show in this short section that probabilistic stability with safety specifications imply probabilistic reach-avoid-stay specifications in certain sense. Note that the converse side can also be constructed under conditions [23]. We omit this part considering the theme of this paper.

The following proposition states the property that if a closed set A is Pr-U.S. for \mathcal{S}_δ , then any weak solutions starting at x from a compact subset of the domain of attraction with probability p is uniformly attracted to A with probability at least p .

Proposition 3.1: Suppose that a closed set $A \subset \mathcal{D}$ is Pr-U.S. for \mathcal{S}_δ . Let K be a compact set and $p \in (0, 1)$. Then the following two statements are equivalent:

- (1) For any solution $X \in \bigcup_{x \in K} \Phi_\delta(x, W)$,

$$\mathcal{P}^X \left[\lim_{t \rightarrow \infty} |X_{t \wedge \sigma}|_A = 0 \right] \geq p.$$

- (2) For every $r > 0$, there exists $T = T(r, \varepsilon)$ such that for any $X \in \bigcup_{x \in K} \Phi_\delta(x, W)$,

$$\mathcal{P}^X [|X_{t \wedge \sigma}|_A < r, \ \forall t \geq T] \geq p.$$

By adding the condition $\{\sigma = \infty\}$ to Proposition 3.1 and using the definitions of the two specifications, we immediately obtain the following connection.

Corollary 3.2: If \mathcal{S}_δ satisfies a stability with safety guarantee specification (\mathcal{X}_0, A, U, p) and \mathcal{X}_0 is compact, then for every $\varepsilon > 0$, \mathcal{S}_δ satisfies the reach-avoid-stay specification $(\mathcal{X}_0, \bar{\mathcal{B}}_\varepsilon(A), U, p)$.

IV. LYAPUNOV-BARRIER CHARACTERIZATION OF PROBABILISTIC STABILITY WITH SAFETY

Recall region \mathcal{D} and \mathcal{R} in Definition 2.9.

Definition 4.1 (Stochastic Lyapunov functions): Let $A \subset \mathcal{D}$ be a closed set. A function $V \in (C^2(\mathcal{B}_R(A)); \mathbb{R}_{\geq 0})$

is said to be a stochastic Lyapunov functions (SLF) w.r.t. A if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$ such that, for all $x \in \mathcal{B}_R(A)$,

$$\alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A) \quad (5)$$

and

$$\sup_{d \in \bar{\mathcal{B}}_\delta} \mathcal{L}_d V(x) \leq -\alpha_3(|x|_A). \quad (6)$$

We first make a quick extension of the existing Lyapunov theorems to systems with point mass perturbations.

Lemma 4.2 (Uniform recurrence): Given an SLF V , there exists some $\eta > 0$ such that, for every $\varepsilon \in (0, 1)$ and $r \in (0, R/2)$, there exists some $T = T(\varepsilon, \eta, r) > 0$ such that $\inf_{X \in \Phi_\delta(x, W)} \mathcal{P}^X[\tau < T] \geq 1 - \varepsilon$, where $\tau = \inf\{t \geq 0 : X_t \in \mathcal{B}_r(A)\}$ is the first hitting time of $\mathcal{B}_r(A)$ for each $X \in \Phi_\delta(x, W)$.

Proposition 4.3: Suppose $A \subset \mathcal{D}$ is compact. If there exists an SLF V w.r.t. A , then A is Pr-U.A.S for \mathcal{S}_δ .

The following result demonstrates that the existence of SLFs is sufficient to characterize probabilistic stability with safety specifications with probabilities depending on initial conditions.

Theorem 4.4: Suppose that $A \subset \mathcal{D}$ is compact and $\mathcal{B}_R(A) \subset \mathcal{R}$. If there exists an SLF $V \in (C^2(\mathcal{B}_R(A)); \mathbb{R}_{\geq 0})$ and some $G := \mathcal{B}_r(A)$ such that (i) $r \in (0, R]$ and $G \subset \mathcal{D}$; (ii) $\mathcal{X}_0 \subset G$, then \mathcal{S}_δ satisfies the probabilistic stability with safety specification $(\mathcal{X}_0, A, U, 1 - \frac{\sup_{x \in \mathcal{X}_0} V(x)}{\alpha_1(r)})$.

Lemma 4.5: For each $X \in \bigcup_{x \in \mathcal{X}_0} \Phi_\delta(x, W)$, set $\tau := \inf\{t \geq 0 : X_t \in G^c\}$. Then for all $X \in \bigcup_{x \in \mathcal{X}_0} \Phi_\delta(x, W)$,

$$\mathcal{P}^X[\lim_{t \rightarrow \infty} |X_t|_A = 0 \mid \tau = \infty] = 1.$$

Remark 4.6: Lemma 4.5 shows that SLFs eliminate the possibility of safe sample paths up/down-crossing any neighborhood of A infinitely often. [13, Theorem 2, Chap II] demonstrates the same result by constructing the total time spent in $G \setminus \mathcal{B}_\varepsilon(A)$ after time t and showing that it converges a.s. to 0 as $t \rightarrow \infty$.

Note that that conditions $\alpha_1(x) \leq V(x)$ and $\sup_{d \in \bar{\mathcal{B}}_\delta} \mathcal{L}_d V \leq 0$ play the role of guaranteeing the probabilistic set invariance. We refer these conditions as the stochastic barrier certificates. An application in control synthesis, termed as stochastic control barrier functions, has been shown in [31, Proposition III.8] with better safety probability compared to the zeroing-type barrier certificates [25], however, less effectiveness than the reciprocal-type barrier certificates (see the definition in [31, Definition III.1]). A thorough comparison between the above mentioned stochastic barrier functions can be found in [31]. To provide stability with safety with a higher probability, one can combine SLF with the reciprocal-type barrier functions.

Theorem 4.7: Under the same assumption in Theorem 4.4. Suppose there exists an SLF $V \in (C^2(\mathcal{B}_R(A)); \mathbb{R}_{\geq 0})$, some $G := \mathcal{B}_r(A)$ such that $G \in \mathcal{D}$ and $\mathcal{X}_0 \subset G$, as well as a function $B \in (C^2(G); \mathbb{R}_{\geq 0})$ satisfying

(i) $\exists \alpha_1, \alpha_2 \in \mathcal{K}$ s.t.

$$\frac{1}{\alpha_1(|x|_A)} \leq B(x) \leq \frac{1}{\alpha_2(|x|_A)}, \quad \forall x \in G; \quad (7)$$

(ii) $\exists \alpha_3 \in \mathcal{K}$ s.t.

$$\sup_{d \in \bar{\mathcal{B}}_\delta} [\mathcal{L}_d B(x) - \alpha_3(|x|_A)] \leq 0, \quad \forall x \in G. \quad (8)$$

Then \mathcal{S}_δ satisfies the probabilistic stability with safety specification $(\mathcal{X}_0, A, U, 1)$.

Remark 4.8: Compared to Theorem 4.4, for the same p , the set G in Theorem 4.7 already covers a larger feasible region within the safe set and provides a milder condition on \mathcal{X}_0 with the help of function B . If $U^c := \{x \in \mathbb{R}^n : h(x) \geq 0\}$ where h is smooth, one can possibly enlarge G such that $G \cap U \neq \emptyset$ with $\partial(G \cap U)$ being piecewise smooth. To see the satisfaction of stability with safety specifications, along with the old conditions, one can introduce an extra reciprocal-type barrier function, denoted by \tilde{B} , and verify extra conditions that are similar to (7) and (8) by replacing $|x|_A$ with $h(x)$.

V. APPLICATIONS IN CONTROL PROBLEMS

In this section, based on the results from Section III and IV, we make a straightforward extension to a stochastic control Lyapunov-barrier characterization for \mathcal{S}_δ satisfying a probabilistic reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, p)$ under controls. As a continuation of [22], we conduct a case study on enhancing the performance of jet engine compressors, under both noisy disturbances and bounded point mass perturbations, based on a reduced Moore-Greitzer nonlinear SDE model.

A. Probabilistic reach-avoid-stay control via stochastic control Lyapunov-barrier functions

We first recast the notion from Section II for control systems. Given a nonempty compact convex set of control inputs $\mathcal{U} \subset \mathbb{R}^p$, consider a nonlinear system of the form

$$dX_t = f(X_t)dt + \xi(t)dt + b(X_t)u dt + g(X_t)dW_t, \quad X_0 = x, \quad (9)$$

where the mapping $b : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$ is smooth; $u : \mathbb{R}_{\geq 0} \rightarrow \mathcal{U}$ is a locally bounded measurable control signal, whilst the other notation remains the same.

Definition 5.1 (Control strategy): A control strategy is a function

$$\kappa : \mathbb{R}^n \rightarrow \mathcal{U}. \quad (10)$$

We further denote $\mathcal{S}_\delta^\kappa$ by the control system driven by (9) that is comprised by $u = \kappa(x)$.

Definition 5.2: (Probabilistic reach-avoid-stay control-lable): Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$ and $p \in [0, 1]$, \mathcal{S}_δ is said to be probabilistic reach-avoid-stay controllable w.r.t. $(\mathcal{X}_0, \Gamma, U, p)$, if there exists a Lipschitz continuous control strategy κ such that the system $\mathcal{S}_\delta^\kappa$ satisfies the specification $(\mathcal{X}_0, \Gamma, U, p)$.

Proposition 5.3: Given $\mathcal{X}_0, \Gamma \subseteq \mathcal{D}$, if there exists a smooth function $V \in (C^2(\mathcal{B}_R(A)); \mathbb{R}_{\geq 0})$ and some $G := \mathcal{B}_r(A) \subset S$, such that

(i) $r \in (0, R]$, $G \subset \mathcal{D}$ and $\mathcal{X}_0 \subset G$;

(ii) $\alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A)$ and

$$\inf_{u \in \mathcal{U}} \sup_{x \in S} \sup_{d \in \delta \mathcal{B}} [\mathcal{L}_d^u V(x) + \alpha_3(|x|_A)] \leq 0,$$

for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}$, where $\mathcal{L}_d^u V(x) := \mathcal{L}_d V(x) + \nabla V(x) \cdot b(x)u$.

Then \mathcal{S}_δ is probabilistically reach-avoid-stay controllable w.r.t. $(\mathcal{X}_0, \Gamma, U, 1 - \frac{\sup_{x \in \mathcal{X}_0} V(x)}{\alpha_1(r)})$.

Remark 5.4: Similarly, one can extend the above proposition to find sufficient conditions for a ‘probability 1’ reach-avoid-stay based on Theorem 4.7. Apart from the conditions in Proposition 5.3, one need to additionally verify if there exists a $B \in (C^2(G); \mathbb{R}_{\geq 0})$ satisfying

- (i) $\frac{1}{\tilde{\alpha}_1(|x|_A)} \leq B(x) \leq \frac{1}{\tilde{\alpha}_2(|x|_A)}$, $\forall x \in G$ for some class- \mathcal{K} functions $\tilde{\alpha}_1, \tilde{\alpha}_2$;
- (ii) $\inf_{u \in \mathcal{U}} \sup_{x \in S} \sup_{d \in \delta \mathcal{B}} [\mathcal{L}_d^u B(x) - \tilde{\alpha}_3(|x|_A)] \leq 0$, for some class- \mathcal{K} function $\tilde{\alpha}_3$.

Remark 5.5: In view of Remark 4.8, the region G can be further relaxed if ∂U is smooth enough. Correspondingly, some extra conditions given by another reciprocal control barrier function are needed to guarantee the sufficiency of $(\mathcal{X}_0, \Gamma, U, 1)$ controllability.

B. Case study

We use the reduced Moore-Greitzer SDE model with an additive control input $[v, 0]^T$ and a multiplicative noise to illustrate the effectiveness. The model is given as:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \Phi(t) \\ \Psi(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{l_c} (\psi_c - \Psi(t)) \\ \frac{1}{16l_c} (\Phi(t) - \mu \sqrt{\Psi(t)}) \end{bmatrix} + \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \\ &+ \varepsilon \begin{bmatrix} (\Phi(t) - \Phi_e(\mu))\beta_1(t) \\ (\Psi(t) - \Psi_e(\mu))\beta_2(t) \end{bmatrix} + \begin{bmatrix} v(t) \\ 0 \end{bmatrix}, \end{aligned} \quad (11)$$

where $\psi_c = a + \iota * [1 + \frac{3}{2}(\frac{\Phi}{\Theta} - 1) - \frac{1}{2}(\frac{\Phi}{\Theta} - 1)^3]$, β_1, β_2 are i.i.d. Brownian motions, $(\Phi_e(\mu), \Psi_e(\mu)) =: X_e(\mu)$ are equilibrium points for $\xi_1, \xi_2, v \equiv 0$. The other parameters are as follows:

$$l_c = 8, \iota = 0.18, \Theta = 0.25, a = 0.67\iota, \varepsilon = 0.08, \delta = 0.01.$$

The physical meanings of variables, parameters and the description of this model can be found in [22, Section V].

Remark 5.6: For $\xi_1, \xi_2, v \equiv 0$, the system admits a family of equilibrium points $X_e(\mu)$ depending on the tunable parameter μ . As μ drops in the neighborhood of the deterministic Hopf bifurcation point, the system undergoes a D-bifurcation (the stability of the invariant measure $\delta_{\{X_e\}}$ changes and a new invariant measure in $\mathbb{R}^n \setminus \{X_e\}$ is built up) and a P-bifurcation (the shape of density of the new measure changes). The full stochastic Hopf bifurcation diagram in [2, Fig 9.13] conveys the brief idea.

Within the a.s. exponentially stable region, any bounded perturbation ξ causes a bounded long-term perturbation of $X_e(\mu)$, and ultimately formulate a compact set containing $X_e(\mu)$. For unstable $\delta_{\{X_e\}}$, especially for those after P-bifurcation, we are interested in stabilizing the robust system to a compact set.

Problem 5.7: We aim to manipulate μ and v simultaneously such that the state (Φ, Ψ) are regulated to satisfy reach-avoid-stay specification $(\mathcal{X}_0, \Gamma, U, 1)$. We require that

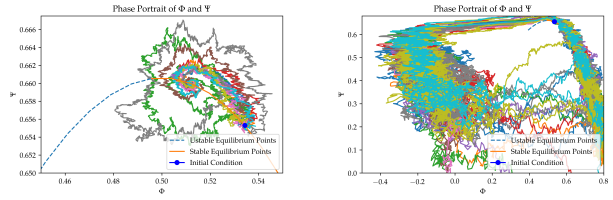


Fig. 1. Given the initial condition of $(\Phi_0, \Psi_0) = (0.5343, 0.6553)$, the phase portraits are generated by (11) with $\mu = 0.63$ (left) and $\mu = 0.59$ (right) under $d_1, d_2, v \equiv 0$.

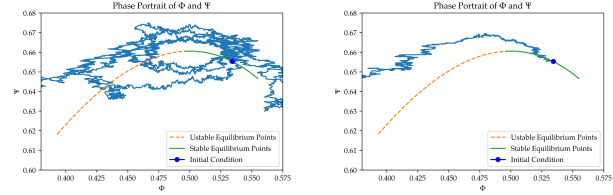


Fig. 2. For $(\Phi_0, \Psi_0) = (0.5343, 0.6553)$, two sample paths (left: converges; right: diverges) are generated with $\mu = 0.609$ (in between the 1st D-bifurcation point and P-bifurcation point) under $d_1, d_2, v \equiv 0$.

$\mu : \mathbb{R}_{\geq 0} \rightarrow [0.5, 1]$ is time-varied with $\mu(0) \in [0.62, 0.66]$ and $|\mu(t + \tau) - \mu(t)| \leq 0.01\tau$ for any $\tau > 0$. We define $\mathcal{X}_0 = \{(\Phi_e(\mu(0)), \Psi_e(\mu(0)))\}$; Γ to be the ball that centered at $\gamma = (0.4519, 0.6513)$ with radius $r = 0.013$, i.e. $\Gamma = \gamma + r\bar{\mathcal{B}}$; the unsafe set $U = \{(x, y) : h_1 \leq 0\} \cap \{(x, y) : h_2 \leq 0\}$, where $h_1(x, y) = -|(x, y) - (0.49, 0.64)| + 0.055$, $h_2(x, y) = |(x, y) - (0.50, 0.65)| - 0.003$. We set $v \in \mathcal{U} = [-0.05, 0.05] \cap \mathbb{R}$.

We refer readers to [22, Remark 30] for treatments of μ as another control input. For each SDE, the signals ξ_1, ξ_2 of each sampling time is generated randomly from $\{-0.1, 0.1\}$.

We choose SLF $V(x, y) = \frac{l_c}{2}(x - \gamma_1)^2 + 8l_c(y - \gamma_2)^2$ and $\alpha_3(x) = 0.1x$; set $B_i = -\log\left(\frac{h_i}{1+h_i}\right)$ for $i = 1, 2$. The settings for the quadratic programming keep the same as [22, Section V.B]. We mix sample paths under different ξ_1, ξ_2 and show the simulation results as in Fig. V-B.

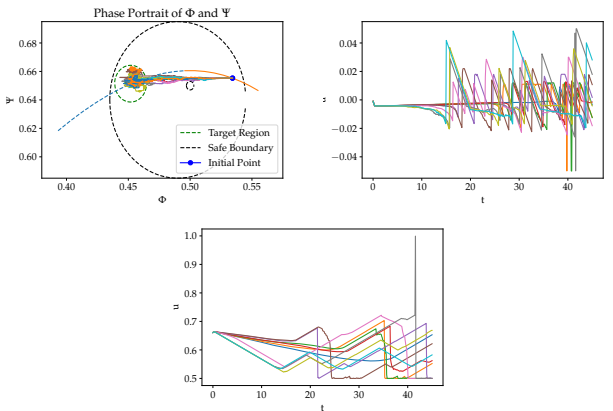


Fig. 3. Controlled sample paths, control input v, μ of Problem 5.7.

Remark 5.8: Note that we have adopted reciprocal type barrier functions, which potentially generates impulse-like

control signals (to cancel the diffusion effects) and terminates the programming. However, once the synthesis succeeds, the feasible controlled sample paths satisfy the specification.

VI. CONCLUSIONS

In this paper, we formulated stochastic Lyapunov-barrier functions to develop sufficient conditions on probabilistic reach-avoid-stay specifications. Given uncertainties of the model, robustness was taken into account such that a worst-case scenario is guaranteed. We characterized a general topological structure of the initial sets, target sets and unsafe sets under the stochastic settings and discussed relaxations given the smoothness of the unsafe boundary. We investigated the effectiveness in a case study of jet engine compressor control problem. Despite of the potentially unbounded control inputs, the control version of SLF along with reciprocal-type barrier functions guarantee a probability-1 satisfaction.

However, just like deterministic Lyapunov-like functions only providing a stability characterization of the solutions, the stochastic Lyapunov-type argument can only estimate a lower bound of ‘satisfaction in probability/law’ without solving the evolving states and distributions. It renders more difficulties of selecting Lyapunov/barrier functions under the restrictive geometric requirements of the initial conditions and unsafe sets.

For future work, compared to the rough estimation of ‘probabilistic domain of satisfactions’ given Lyapunov-like functions, it would be necessary to consider accurate evaluation of laws and investigate formal methods in providing more reliable schemes on finding the probabilistic winning sets (from which the specifications are satisfied). Considering the uncertainties of stochastic modelling, to provide soundness and (possibly weak) completeness, stochastic abstraction analysis is fundamental to the robust stochastic control synthesis problems.

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