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Control of Nonlinear Systems with Reach-Avoid-Stay Specifications: A Lyapunov-Barrier Approach with an Application to the Moore-Greizer Model

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Abstract— **The study of control synthesis for reach-avoid-stay objectives for nonlinear systems has received considerable interest in recent years. Such objectives can be naturally treated as a formal specification and effectively handled by formal methods. While formal methods often rely on constructing a finite-state approximation and developing algorithms to capture the winning set (a set of initial states from which a controller exists to realize the given task), Lyapunov methods can characterize stability and safety properties without having to discretize the state space. Inspired by recent work on converse Lyapunov-barrier theorems, we propose control Lyapunov-barrier functions to provide sufficient conditions for control synthesis with reach-avoid-stay specifications. A comparison between the proposed Lyapunov method and formal methods based on a fixed-point algorithm is illustrated by an application to enhancing the performance of jet engine compressors, which is based on a reduced Moore-Greitzer nonlinear ODE model. We apply a quadratic programming (QP) framework to reactively synthesize controllers in the case study.**

Index Terms— **Reach-avoid-stay specifications, control Lyapunov-barrier functions, formal methods, fixed-point algorithm, Moore-Greitzer nonlinear ODE model, quadratic programming.**

I. Introduction

Control synthesis of reach-avoid-stay specifications is concerned with finding control strategies to regulate the state of a dynamical system such that the trajectory always avoids an unsafe set whilst the state reaches a target set within a finite time and stays inside it afterwards. In applications to a variety of safety-critical control problems, such as robotic motion planning and regulation of trajectory [9], [7], [22], control of reach-avoid-stay specifications has received a surge of interests.

The reach-avoid-stay property is one of the building blocks in temporal logics for specifying more complex task objectives. The advent of various temporal logic languages and the corresponding model checking algorithms [4] made it possible to verify and synthesize controllers w.r.t. linear temporal properties for finite-transition systems [22], [15], [26]. Formal methods for nonlinear systems rely on a finite abstraction (or symbolic model) of the original systems, based on which computational methods are developed [11], [18]. Aside from the abstraction analysis and the computational complexity caused by state space discretization, formal methods in control synthesis computes with guarantees a set of initial states from which a controller exists to realize the given specification [5]. In [17], a fixed-point algorithm was developed for reach-and-stay specification regarding the computation complexity issue by adaptively partitioning the state space. However, for systems that also depend on tunable parameters and undergo bifurcations, such as the Moore-Greitzer model for jet engine compressors studied in this paper, there are two challenges for using formal methods: 1) the sampling time for constructing abstractions is highly related to the parameters since the system state evolves with different rates for different parameters, and 2) there are constraints on the change rates of the tunable parameters that can be treated as control inputs. Formal control synthesis tools such as [24], [16] cannot be used readily to design control strategies for such systems because of these challenges.

In contrast to formal methods, Lyapunov-like functions, such as control Lyapunov functions (CLF) and control barrier functions (CBF), are able to provide feedback stabilizing and feedback set-invariance related controllers without state-space discretization and considering local dynamics [1], [2], [14], [21]. In addition, since asymptotic stability entails asymptotic attraction, reachability can be naturally captured by asymptotic stability properties. In [2], the authors provided a quadratic programming framework for uniting the Lyapunov and barrier certificates to reactively synthesize optimal control signals for safety and stabilization objectives, however, without guarantees due to the possible conflict choices of target sets and unsafe sets. Recent works [23] derived sufficient conditions for stabilization with safety guarantees by uniting Lyapunov and barrier functions. Such conditions, however, provide unnatural topological structure on the unsafe set [6] (see also Remark 21 of this paper). Inspired by the work [27], the authors of [20] have recently shown that it is possible to construct Lyapunov functions that is defined on the entire set of initial conditions from which stabilization with safety guarantees is achievable. The recent work [19] also provided a perspective of formulating barrier function via Lyapunov functions. This idea was used in [20] to establish a connection between stabilization with safety guarantees and reach-avoid-stay specifications.

Motivated by the work in [20], which only deals with converse Lyapunov-barrier functions for systems without control, this paper formulates control Lyapunov-barrier functions that can be used to obtain sufficient conditions for

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developing state-dependent control signals w.r.t. the reach-avoid-stay specifications. In contrast to the work of [23] and [2], we provide more general choices of the initial, unsafe, and target sets, which makes it more flexible to design control Lyapunov-barrier functions in practice.

The rest of this paper is organized as follows. In Section II, we present the problem definition. In Section III, we convert the system into a stabilize-avoid-stay problem with soundness guarantees and formally define control Lyapunov-barrier functions regarding that problem. In Section IV, we design control Lyapunov-barrier functions and provide sufficient conditions on the control values. In section V, a case study on a Moore-Greitzer model is conducted to demonstrate a comparison between Lyapunov methods and formal methods. The paper is concluded in Section VI.

Notation: We denote the Euclidean space by $\mathbb{R}^n$ for $n > 1$. The ball of radius r centered at x by $x + r\mathcal{B} = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$, where $\|\cdot\|$ is the Euclidean norm. We denote $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. Given two sets $A, B \subset \mathbb{R}^n$, the set difference of B and A is defined by $B \setminus A = \{x \in B : x \notin A\}$. For a closed set $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote the distance from x to A by $\|x\|_A = \inf_{y \in A} \|x - y\|$ and r-neighborhood of A by $A + r\mathcal{B} = \bigcup_{x \in A}(x + r\mathcal{B})$.

II. Problem Definition

Given a nonempty compact set of control inputs $\mathcal{U} \subset \mathbb{R}^p$, consider a nonlinear system of the form

$$x'(t) = f(x(t)) + g(x(t))u(t),\ x(0) = x_0, \tag{1}$$

where $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the state trajectory; $u : \mathbb{R}_{\geq 0} \to \mathcal{U}$ is a bounded control signal; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz nonlinear vector field; the mapping $g : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ is smooth.

We denote a set of control signals (possibly generated from a close-loop control strategy as defined below) by $\mathbf{u}$, denote the control system driven by (1) under $\mathbf{u}$ by $\mathcal{S}_\mathbf{u}$, and denote a solution of $\mathcal{S}_\mathbf{u}$, starting from x_0 under a control signal $u \in \mathbf{u}$, by $x_u(t, x_0)$. For simplicity, we also denote the solution by $x_u(t)$ if x_0 is not emphasized. The set of all solution for $\mathcal{S}_\mathbf{u}$ starting from x_0 is denoted by $\Phi_\mathbf{u}(x_0)$.

Definition 1 (Reachable set)**.** *The reachable set of system $\mathcal{S}_\mathbf{u}$ at time $t \geq 0$ starting from $W \subset \mathbb{R}^n$ is defined by*

$$\mathcal{R}^t_\mathbf{u}(W) = \{x_u(t) : x_u(\cdot) \in \Phi_\mathbf{u}(x_0), x_0 \in W\}. \tag{2}$$

Furthermore, for $T \geq 0$, we define

$$\mathcal{R}^{t \geq T}_\mathbf{u}(W) = \bigcup_{t \geq T} \mathcal{R}^t_\mathbf{u}(W), \quad \mathcal{R}_\mathbf{u}(W) = \bigcup_{t \geq 0} \mathcal{R}^t_\mathbf{u}(W). \tag{3}$$

Definition 2 (Controlled invariant set)**.** *A set $\Omega \subseteq \mathbb{R}^n$ is said to be a controlled invariant set if for all $x(0) \in \Omega$ there exists a nonempty collection $\mathbf{u}$ of control signals, such that solutions $\Phi_\mathbf{u}(x_0)$ are well defined, and $\mathcal{R}_\mathbf{u}(\Omega) \subseteq \Omega$.If the above conditions hold, we also say that Ω is controlled invariant for $\mathcal{S}_\mathbf{u}$.*

Definition 3 (Controlled safe set)**.** *Given an unsafe set $U \subset \mathbb{R}^n$, a set $W \subset \mathbb{R}^n$ for is said to be controlled safe w.r.t. U if there exists a nonempty collection $\mathbf{u}$ of control signals such that solutions $\Phi_\mathbf{u}(x_0)$ are well defined for all $t \geq 0$, and $\mathcal{R}_\mathbf{u}(W) \cap U = \emptyset$. If the above conditions hold, we also say that W is controlled safe w.r.t. U for $\mathcal{S}_\mathbf{u}$.*

Definition 4 (Reach-avoid-stay specification)**.** *A system $\mathcal{S}_\mathbf{u}$ satisfies a reach-avoid-stay specification (W, U, Ω), where $W, U, \Omega \subseteq \mathbb{R}^n$, if the following conditions hold:*

(i) *(safe w.r.t. U) W is a controlled safe set w.r.t. U for $\mathcal{S}_\mathbf{u}$;*
(ii) *(reach-and-stay w.r.t. Ω) there exists a finite reachable time $T \geq 0$ such that $\mathcal{R}^{t \geq T}_\mathbf{u}(W) \subseteq \Omega$.*

Definition 5 (Control strategy)**.** *A control strategy is a function*

$$\kappa : \mathbb{R}^n \to 2^{\mathcal{U}}. \tag{4}$$

Definition 6 (State-dependent control signal)**.** *A control signal u is said to conform to a control strategy κ if*

$$u(t) \in \kappa(x(t)), \quad \forall t \geq 0. \tag{5}$$

We denote the set of all control signals that conform to κ by $\mathbf{u}_\kappa$.

Problem 7 (Reach-avoid-stay control)**.** *Given a reach-avoid-stay specification (W, U, Ω), design a control strategy κ such that the resulting solutions of $\mathcal{S}_{\mathbf{u}_\kappa}$ satisfy (W, U, Ω).*

Since reachability is closely related set stability, we also define stability w.r.t. a closed set.

Definition 8. *(Set stability) A closed set $A \subseteq \mathbb{R}^n$ is said to be uniformly asymptotically stable (U.A.S.) if there exists a nonempty set $\mathbf{u}$ of control signals, such that for any $u \in \mathbf{u}$ the solution $x_u(t)$ is defined for $t \geq 0$ (forward complete) and the following two conditions hold:*

(i) *(uniform stability) for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that $\|x_u(t)\|_A < \varepsilon$ whenever $\|x_u(0)\|_A < \delta_\varepsilon$;*
(ii) *(uniform attractivity) there exists some $\rho > 0$ such that for every $\varepsilon > 0$, there exists some $T > 0$ such that whenever $\|x_u(0)\|_A < \rho$ and $t \geq T$ the solution $\|x_u(t)\|_A < \varepsilon$.*

If the above conditions hold, we also say that the set A is U.A.S. for $\mathcal{S}_\mathbf{u}$.

Definition 9 (Domain of attraction)**.** *Given a set $\mathbf{u}$ of control signals, if a closed set $A \subseteq \mathbb{R}^n$ is U.A.S. under $\mathcal{S}_\mathbf{u}$, we correspondingly define the domain of attraction of A, denoted by $\mathcal{G}_u(A)$, as the set of all initial states $x \in \mathbb{R}^n$ such that $x_u(t)$ converges to the set A for any $u \in \mathbf{u}$, equivalently we have*

$$\mathcal{G}_\mathbf{u}(A) = \{x \in \mathbb{R}^n : \lim_{t \to \infty} \|x_u(t)\|_A = 0\ \forall u \in \mathbf{u}\}. \tag{6}$$

III. Reach-Avoid-Stay Control Design via Lyapunov Method

In this section, we propose sufficient certificates on the control strategy κ to be designed for Problem 7. In cooperating with Lyapunov like functions, we first convert the reach-avoid-stay specification into a stronger objective, such that the satisfaction of the latter case will guarantee the reach-avoid-stay property.

Definition 10 (Stability with safety guarantee specification). *A system $\mathcal{S}_{\mathbf{u}}$ satisfies a stability with safety guarantee specification (W,U,A), where $W,U,A \subseteq \mathbb{R}^n$, if the following conditions hold:*

(i) *the set A is closed and U.A.S. for $\mathcal{S}_{\mathbf{u}}$;*
(ii) *$W \in \mathbb{R}^n$ is a controlled safe set w.r.t. U for $\mathcal{S}_{\mathbf{u}}$;*
(iii) *$W \subseteq \mathcal{G}_{\mathbf{u}}(A)$.*

Proposition 11. *Let $\mathbf{u}$ be a nonempty set of control signals $\mathbf{u}$ such that $\mathcal{S}_{\mathbf{u}}$ satisfies a stability with safety guarantee specification (W,U,A) for a compact W. Then for every $\varepsilon > 0$, $\mathcal{S}_{\mathbf{u}}$ also satisfies the reach-avoid-stay specification $(W,U,A+\varepsilon\mathcal{B})$.*

The proof of Proposition 11 relies on the following lemma. The detailed proof can be found in [19, Proposition 25].

Lemma 12 (Uniformity of attraction). *Suppose that a closed set $A \subset \mathbb{R}^n$ is uniformly stable for $\mathcal{S}_{\mathbf{u}}$. Let K be a compact set. Then the following two statements are equivalent:*

1) *For any $x_0 \in K$ and any $x_u(t) \in \Phi_{\mathbf{u}}(x_0)$, $x_u(t)$ is defined for all $t \geq 0$ and*
$$\lim_{t\to\infty} \|x_u(t)\|_A = 0.$$
2) *For every $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that*
$$\|x_u(t)\|_A < \varepsilon$$
holds for any $x_0 \in K$, $x_u(t) \in \Phi_{\mathbf{u}}(x_0)$, and $t \geq T$.

Proof of Proposition 11 Since the set A is U.A.S., for any controlled safe set W w.r.t. U for $\mathcal{S}_{\mathbf{u}}$, we can show $W \subseteq \mathcal{G}_{\mathbf{u}}(A)$. If W is compact, by Lemma 12, for any $u \in \mathbf{u}$, there exists a finite time $T > 0$ such that $\|x_u(t)\|_A \leq \varepsilon$ whenever $t \geq T$ and $x_0 \in W$. The above argument is equivalent as $\mathcal{R}_{\mathbf{u}}^{t\geq T(\varepsilon)}(W) \subset A+\varepsilon\mathcal{B}$ for any $u \in \mathbf{u}$, which justifies the reach-and-stay specification w.r.t. $A+\varepsilon\mathcal{B}$. The safe specification automatically hold based on (ii) of Definition 10. ■

Problem 13 (Stabilization with safety control). *Given a stabilization with safety specification (W,U,A), design a control strategy κ such that the resulting solutions of $\mathcal{S}_{\mathbf{u}_\kappa}$ satisfies (W,U,A) .*

Corollary 14. *Let $A+\varepsilon\mathcal{B} \subseteq \Omega$ for some $\varepsilon > 0$. The solution to Problem 13 also solves Problem 7.*

Proof It is straightforward from Proposition 11. ■

For the rest of this paper, we focus on deriving sufficient conditions to solve Problem 13. Before proceeding, it is necessary to introduce control Lyapunov functions to interpret the U.A.S. property, as well as control barrier certificates to separate the invariant and unsafe regions.

A. Preliminaries on control Lyapunov and barrier functions

Definition 15 (Control Lyapunov function w.r.t. a closed set A). *Let $D \subset \mathbb{R}^n$ be an open set containing a closed set $A \subset \mathbb{R}^n$. A continuously differentiable function $V : D \to \mathbb{R}_{\geq 0}$ is said to be a control Lyapunov function (CLF) w.r.t. A if*

(i) *there exist class $\mathcal{K}$ functions α_1 and α_2 such that*
$$\alpha_1(\|x\|_A) \leq V(x) \leq \alpha_2(\|x\|_A) \tag{7}$$
for all $x \in D$;
(ii) *there exists a class $\mathcal{K}$ function α_3 such that for all $x \in \{z \in D \setminus A : L_gV(z) = 0\}$,*
$$L_fV(x) \leq -\alpha_3(\|x\|_A) \tag{8}$$
holds.

Definition 16. *Given a control Lyapunov function V w.r.t. a closed set A, we further define a state-dependent control strategy*
$$\upsilon(x) := \{u \in \mathcal{U} : L_fV(x) + L_gV(x)u \leq -\alpha_3(\|x\|_A)\}. \tag{9}$$

Remark 17. *Note that when $u \equiv 0$, CLFs are reduced to Lyapunov functions. If there exists a control Lyapunov function $V : D \to \mathbb{R}_{\geq 0}$ w.r.t. a closed set $A \subset D$ and the corresponding set of control values $\upsilon(x)$, it can be justified that for all $x_0 \in D$, if $u(t) \in \upsilon(x_u(t))$ for all $t \geq 0$, the set D is controlled invariant for $\mathcal{S}_{\mathbf{u}_\upsilon}$(see in Definition 6), and the closed set A is U.A.S. for $\mathcal{S}_{\mathbf{u}_\upsilon}$.*

Remark 18. *As a special case, set $A = \{0\}$, $D = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^p$, Sontag in [25] provided a universal feedback control strategy $\kappa : \mathbb{R}^n \to \mathcal{U}$ by*
$$\kappa(x) = \begin{cases} -\frac{L_fV(x)+\sqrt{(L_fV(x))^2+(L_gV(x))^4}}{L_gV(x)} & \text{if} \quad L_gV(x) \neq 0; \\ 0 & \text{if} \quad L_gV(x) = 0. \end{cases} \tag{10}$$
It is clear that $u = \kappa(x) \subset \mathbf{u}_\upsilon$ renders the system (globally) asymptotically stable at $\{0\}$. However, the special form κ is usually not compatible with the safety control strategy.

Definition 19 (Control barrier function). *Given sets $C,U \subset \mathbb{R}^n$, a continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ is said to be a control barrier function (CBF) for (C,U) if the following conditions hold:*

(i) *$B(x) \geq 0$ for all $x \in C$;*
(ii) *$B(x) < 0$ for all $x \in U$;*
(iii) *for all $x \in \{z : B(z) = 0$ and $L_gB(z) = 0\}$, $L_fB(x) > 0$.*

Remark 20. *Similar to (9), we can find a state-dependent control strategy*
$$\beta(x) := \{u \in \mathcal{U} : L_fB(x) + L_gB(x)u > 0\ \forall x \text{ s.t. } B(x) = 0\},$$
then for all $x_0 \in C$, if $u(t) \in \beta(x_u(t))$ whenever $t \in \{\tau \geq 0 : B(x(\tau)) = 0\}$, the set C is controlled invariant for $\mathcal{S}_{\mathbf{u}_\beta}$.

Remark 21. *Note that a control barrier function B works in a way to separate two disjoint sets C and U, and meanwhile guarantees the invariance of C. However, the sets C and U are free to choose as long as they are disjoint, e.g. it is not necessary that $\overline{C} \cap \overline{U} \neq \emptyset$ or $C \cup U = \mathbb{R}^n$. Furthermore, to deal with safety control problems, it is natural to arbitrarily look for a set C containing the initial set of states W and conversely find a CBF for (C,U). For the situations when the initial set $W \not\subset C \cup U$, one needs extra conditions to keep the trajectory away from U. In [23], this situation was* *discussed. The extra condition was given as $\overline{\mathbb{R}^n \setminus (C \cup U)} \cap \overline{U} = \emptyset$.However, this condition restricted the topology in the following ways. In [6], it has been justified that the unsafe set U in unbounded. In [20], it is shown that $x \in \partial U \Rightarrow B(x) = 0$, i.e. $\overline{C} \cap \overline{U} \neq \emptyset$.*

Inspired by the role of CLF and CBF, we propose control Lyapunov-barrier functions as well as the corresponding state-dependent control strategies to guarantee a stabilization with safety guarantee specification (W,U,A).

B. Stabilization with safety guarantees

Since the set Ω is given in a priori, we can arbitrarily find a closed set A such that $A + \varepsilon\mathcal{B} \subseteq \Omega$ for some $\varepsilon > 0$ and specify an unsafe set U such that $U \cap \Omega = \emptyset$. We require an initial set of states W to be compact and $W \cap U = \emptyset$. For Problem 13 (stabilization with safety control (W,U,A)), we propose the following set of sufficient conditions on control strategies by combining the notions of CLF and CBF.

Theorem 22. *Given a closed set A and an unsafe set U such that $A \cap U = \emptyset$, assume there exists an open set D such that $(A \cup W) \subset D$, a control Lyapunov function $V : D \to \mathbb{R}_{\geq 0}$ w.r.t. A, as well as a continuously differentiable functions $B : \mathbb{R}^n \to \mathbb{R}$ such that the following conditions hold:*

(i) *$W \subset C := \{x \in D : B(x) \geq 0\}$;*
(ii) *$x \in U \Rightarrow B(x) < 0$;*
(iii) *for all $x \in \{z \in C : L_g B(z) = 0\}$, $L_f B(x) > 0$,*

together with a state-dependent control strategy $\beta(x) = \{u \in \mathcal{U} : L_f B(x) + L_g B(x) u > 0 \; \forall x \in C\}$. Let $\kappa(x) = \upsilon(x) \cap \beta(x)$, then the system $\mathcal{S}_{\mathbf{u}_\kappa}$ satisfies the stabilization with safety guarantees specification (W,U,A).

Proof For system $\mathcal{S}_{\mathbf{u}_\kappa}$, the signal $u \in \mathbf{u}_\kappa$ is such that $u(t) \in \kappa(x_u(t))$ for all $t \geq 0$. We first show the set A is U.A.S. for all $x_0 \in W$ for $\mathcal{S}_{\mathbf{u}_\kappa}$. Indeed, the set D is controlled invariant for $\mathcal{S}_{\mathbf{u}}$. Therefore,

$$\begin{aligned}\dot{V}(x_u(t,x_0)) &= L_f B(x_u(t)) + L_g(B(x_u(t))u(t) \\ &\leq -\alpha_3(\alpha_1^{-1}(V(x_u(t,x_0))),\end{aligned}$$

and for all $x_u(t) \notin A$ we have $\dot{V}(x_u(t,x_0)) < 0$. It can be easily shown that $\|x_u(t)\|_A \xrightarrow{t\to\infty} 0$.

Next we show the controlled-invariant property of the set C. If we suppose the opposite, then there exists some $x_0 \in C$, a signal u such that $u(t) \in \kappa(x_u(t))$ for all $t \geq 0$, and some $s > 0$ such that $B(x_u(s,x_0)) < 0$. We can thereby find the finite $\tau = \sup\{t \geq 0 : x_u(t,x_0) \in C\}$, which makes $B(x_u(\tau,x_0)) = 0$ due to the continuity of $B(x_u(s,x_0))$. Since D is open and $x_u(\tau,x_0) \in D$, we can arbitrarily find a sufficient small $\varepsilon > 0$ such that $x_u(t,x_0) \in D$ for $t \in [\tau, \tau+\varepsilon]$, which implies that

$$\dot{B}(x_u(t,x_0)) = L_f B(x_u(t)) + L_g(B(x_u(t))u(t) > 0, \; \forall t \in [\tau, \tau+\varepsilon]$$

Thus $B(x_u(t,x_0)) > B(x_u(\tau,x_0) = 0$ and $x_u(t,x_0) \in C$ for all $t \in [\tau, \tau+\varepsilon]$. This contradicts with the definition of τ. Hence, the set C is controlled invariant for $\mathcal{S}_{\mathbf{u}_\kappa}$. We also have $\mathcal{R}_{\mathbf{u}_\kappa}(W) \subseteq C$, and therefore $\mathcal{R}_{\mathbf{u}_\kappa}(W) \cap U = \emptyset$. ■

Remark 23. *The idea of the above theorem is to first obtain the region D such that for $x_0 \in W$, the signal u that conform to the strategy υ will lead the trajectory $x_u(t,x_0)$ to asymptotically converge to A. Then we add an extra constraint β on the control values such that W is controlled invariant w.r.t. U. It is easier to combine the notions of CLF and CBF rather than use a single function playing the same role. In [23], a control Lyapunov-barrier function $V : \mathbb{R}^n \to \mathbb{R}$ was introduced to verify the stabilization with safety guarantees (W,U,A), where $A = \{0\}$. The conditions are given as follows:*

(i) *V is lower-bounded and radially unbounded;*
(ii) *$V(x) > 0$ for all $x \in U$;*
(iii) *$L_f V(x) \leq 0$ for all $x \in \{z \in \mathbb{R}^n \setminus \overline{U} : L_g V(z) = 0\}$;*
(iv) *$C = \{x \in \mathbb{R}^n : V(x) \leq 0\}$ and $\overline{\mathbb{R}^n \setminus (C \cup U)} \cap \overline{U} = \emptyset$.*

Note that the verification of W and U based on the sign of a CBF is opposite to the notions used in this paper. However, it can be observed that conditions (i)-(iii) define a CLF w.r.t. A, conditions (iii)-(iv) define a CBF for (C,U). The condition $\overline{\mathbb{R}^n \setminus (C \cup U)} \cap \overline{U} = \emptyset$ is for the situation where C is selected independent of W as discussed in Remark 21. The advantage of verifying the (W,U,A) property using a single CLBF is that the feedback control strategy can be simply given as the form (10). *Howevr, the construction of such CLBF is achieved by separating it into a CLF and a CBF and designing each of them independently.*

If the functions V and B exist, one can obtain a control strategy solving Problem 13, based on which a set of proper constraints on the state-dependent control signals can be obtained. The sufficient conditions on the state-dependent stabilization with safety control signals are as follows:

$$u(t) \in \kappa(x_u(t)), \; \forall t \geq 0. \tag{11}$$

For simplicity, we call the pair (V,B) the control Lyapunov-barrier functions for the stabilize-avoid-stay specification (W,U,A).

IV. Design of Control Lyapunov-Barrier Functions

In this section, we present possible ways of constructing the control Lyapunov-Barrier Functions (V,B) for the stabilization with safety guarantee specification (W,U,A), where A is arbitrarily chosen as a strict subset of Ω. In particular, the set C and D in Theorem 22 should be defined properly w.r.t. V and B. We assume that the sets W, U, Ω, A are given properly.

For stabilization w.r.t. A, the domain D can be selected to be a subset of $\mathbb{R}^n \setminus U$, a candidate control Lyapunov function $V(x)$ can be chosen as

$$V(x) := \sup_{t \geq 0, x \in D, u \in \mathcal{U} \cap \beta(x)} w(x_u(t,x))e^t, \tag{12}$$

where

$$w(x) = \max\left(\|x\|_A, \frac{1}{\|x\|_{\mathbb{R}^n \setminus D}} - \frac{2}{\inf_{x \in A} \|x\|_{\mathbb{R}^n \setminus D}}\right) \tag{13}$$

is a continuous indicator for (A,D), and $w(x)=0$ for $x\in A$; $w(x)>0$ for $x\in D\setminus A$. For any $\{x_m\}\in D$ such that either $x_m\to p\in\partial D$ or $\|x_m\|\xrightarrow{m\to\infty}\infty$, we either have $\|x_m\|_A\to\infty$ or $\frac{1}{\|x_m\|_{\mathbb{R}^n\setminus D}}\to\infty$.

It can be verified that $V:D\to\mathbb{R}_{\geq 0}$ is properly defined and $W\cup A\subseteq D\subseteq\mathcal{G}_{\mathbf{u}_\beta}(A)$. Now the question is reduced to construct B and set C.

Since $C\subseteq D$ does not intersect with the unsafe set U, the function B is to provide barrier certificates and an extra control constraints to force the trajectory starting from $W\subseteq C$ not leaving C. By a (robust) converse barrier function theorem introduced in [19], which is quoted and rephrased as follows, we can find a proper $B(x)$.

Theorem 24 (Safe sets admit robust barrier functions [19])**.** *Suppose sets A and W are compact and $\mathcal{R}_u(W)\cap\overline{U}=\emptyset$. Then there exists a barrier function for (W,U).*

Now we show one of the possibilities.

Proposition 25. *Suppose the assumptions in Theorem 24 hold. Let $V:D\to\mathbb{R}_{\geq 0}$ be a control Lyapunov function w.r.t. A, then there exists a compact C such that $W\cup A\subseteq C$ and $C\cap U=\emptyset$. If $r=\max_{x\in C}(\alpha_2(\|x\|_A))$, where α_2 is the corresponding upper bound for V, and $B(x)=r-V(x)$ satisfies the requirement.*

To prove the above proposition, we need the following lemma to show the topological properties of solutions of $\mathcal{S}_{\mathbf{u}}$, where $\mathbf{u}$ is generated by bounded signals $u\in\mathcal{U}$. It can be found in [8, Theorem 3, Section 7], and it can be automatically recast to the system $\mathcal{S}_{\mathbf{u}}$, since the differential inclusion of $\mathcal{S}_{\mathbf{u}}$ straightforwardly satisfies the basic conditions.

Lemma 26 (Compactness of reachable sets)**.** *Let $K\subseteq\mathbb{R}^n$ be a compact set, let $\mathbf{u}$ be the set of bounded control signal $u\in\mathcal{U}$. Suppose that there exists some $\tau>0$ such that solutions of $\mathcal{S}_{\mathbf{u}}$ starting from K are always defined on $[0,\tau)$. Then, for any $T\in[0,\tau)$, $\mathcal{R}_{\mathbf{u}}^{0\leq t\leq T}(K)$ is a compact set. Furthermore, solutions of $\mathcal{S}_{\mathbf{u}}$ on $[0,T]$ form a compact set under the uniform convergence topology.*

Proof of Proposition 25 By Lemma 12, for any $\rho>0$, we can find a T such that $x_u(t,x_0)\leq\rho$ for all $x_0\in W$ and all $u\in\mathbf{u}_\upsilon$. Since the set D is controlled invariant under such u, it follows that

$$\mathcal{R}_u(W)\subseteq\mathcal{R}_u^{0\leq t\leq T}(W)\cup(A+\rho\mathcal{B})\subseteq D. \tag{14}$$

Since W is compact and $\mathbf{u}_\upsilon\subset\mathbf{u}$, the reachable set $\mathcal{R}_{\mathbf{u}_\upsilon}^{0\leq t\leq T}(W)$ is also compact by Lemma 26. We name $C:=\mathcal{R}_{\mathbf{u}_\upsilon}^{0\leq t\leq T}(W)\cup(A+\rho\mathcal{B})$ and it is clear that C is also compact. If we further have $r=\max_{x\in C}(\alpha_2(\|x\|_A))$, we can justify that $B(x)=r-V(x)$ satisfies requirements, i.e. $B(x)\geq 0$ for all $x\in C$; $x\in U\Rightarrow B_2(x)<0$ (since V is undefined in U); $L_f B_2(x)\geq\alpha_3(\alpha_2^{-1}(r-B_2(x)))\geq 0$ for $B_2(x)\in(-\infty,r]$. ■

Remark 27. *In [2], the authors also provided reciprocal barrier functions. However, The extra condition is to require the invariant set C to be modeled by $C=\{x\in\mathbb{R}^n:h(x)\geq 0\}$, where $h:\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable. Due to the smoothness of h, the interior of C is simply $C^\circ=\{x\in\mathbb{R}^n: h(x)>0\}$. A reciprocal barrier function $B:C^\circ\to\mathbb{R}$ is such that:*

$$\begin{aligned}\frac{1}{\alpha_1(h(x))}\leq B(x)\leq\frac{1}{\alpha_2(h(x))},\\ L_fB(x)+L_gB(x)u\leq\alpha_3(h(x)),\end{aligned} \tag{15}$$

where α_i for $i\in\{1,2,3\}$ are class $\mathcal{K}$ functions.

V. Case Study of Jet Engine Compressor Control

The reduced Moore-Greitzer ODE model is a commonly used nonlinear model for capturing average flow Φ and average pressure Ψ of axial-flow jet engine compressors. As the throttle coefficient μ decreases, surge instability occurs and generates a pumping oscillation (Hopf-bifurcation) that can cause flameout and engine damage [3], [12]. In practice, to deal with the requests from downstream, the operation point needs to be switched during the process. However, without any controls, operation points are determined by μ and a smaller μ may result in unstable operation points [28]. To alleviate the oscillation and prevent substantial pressure loss during the switch, it is motivated to design controllers to lead the state (Φ,Ψ) reach-and-stay in a small region around an unstable operation point, and meanwhile avoid touching the region with low average pressure.

The reduced Moore-Greitzer ODE model with an additive control input $[v,0]^T$ is given as:

$$\frac{d}{dt}\begin{bmatrix}\Phi(t)\\ \Psi(t)\end{bmatrix}=\begin{bmatrix}\frac{1}{l_c}(\psi_c-\Psi(t))\\ \frac{1}{16l_c}\left(\Phi(t)-\mu\sqrt{\Psi(t)}\right)\end{bmatrix}+\begin{bmatrix}v(t)\\ 0\end{bmatrix}, \tag{16}$$

where $\psi_c=a+\iota*[1+\frac{3}{2}(\frac{\Phi}{\Theta}-1)-\frac{1}{2}(\frac{\Phi}{\Theta}-1)^3]$, and for a global parameter set-up, we let

$$l_c=8,\ \iota=0.18,\ \Theta=0.25,\ a=\frac{1}{3.5}.$$

For the system (16) without controls, the operation points are equilibrium points $(\Phi_e(\mu),\Psi_e(\mu))$ of the system, which also depend on the value of μ. Conversely, given an equilibrium point (Φ_e,Ψ_e), $\mu=\mu(\Phi_e,\Psi_e)$. Figure 1 shows the regions of stable/unstable equilibrium points and gives examples of phase portraits w.r.t. stable/unstable equilibrium points.

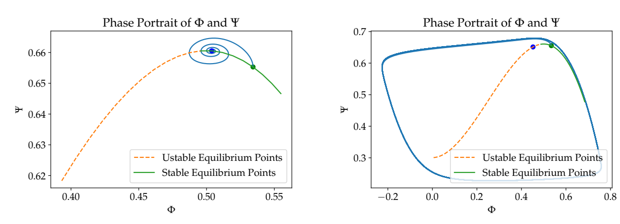

Fig. 1. Regions of stable (green) / unstable (red) equilibrium points. Given the initial condition of $(\Phi_0,\Psi_0)=(0.5343,0.6553)$, the phase portraits are generated by (16) with $\mu=0.62$ (left) and $\mu=0.56$ (right). As μ decreases, the system undergoes a Hopf bifurcation at $\mu\approx 0.6123$.

Remark 28. *The phase portraits in Figure 1 are generated under μ which provide hyperbolic linear spectrum,* *the convergent/divergent rate w.r.t. the equilibrium points is exponential. However, when μ is in a small neighborhood of the bifurcation point ($\mu \approx 0.6123$), the spectrum becomes non-hyperbolic, and the dynamics appear to be slow-varying.*

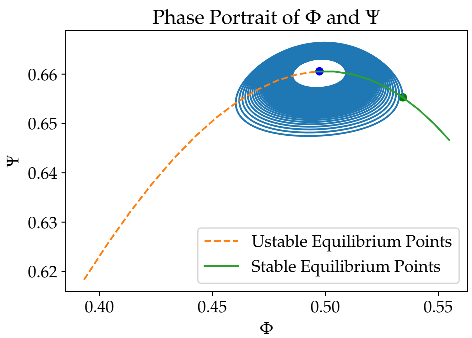

Fig. 2. Phase portrait generated by (16) with $\mu = 0.612$ under the initial condition of $(\Phi_0, \Psi_0) = (0.5343, 0.6553)$.

A. Control problem formulation

Problem 29. *We aim to manipulate the throttle coefficient μ and v simultaneously such that the state (Φ, Ψ) are regulated to satisfy reach-avoid-stay specification (W, U, Ω). We require that $\mu : \mathbb{R}_{\geq 0} \to [0.5, 1]$ is time-varied with $\mu(0) \in [0.62, 0.66]$ and $|\mu(t+\tau) - \mu(t)| \leq 0.01\tau$ for any $\tau > 0$. We define $W = \{(\Phi_e(\mu(0)), \Psi_e(\mu(0)))\}$ (i.e. a sub-region of stable equilibrium points); Ω to be the ball that centered at $\omega = (0.4519, 0.6513)$ with radius $r = 0.003$, i.e. $\Omega = \omega + r\mathcal{B}$; $U = \{(x, y) : x \in (0.497, 0.503), y \in (0.650, 0.656)\}$. We set $v \in \mathcal{U} = [-0.05, 0.05] \cap \mathbb{R}$.*

Addressing Problem 29, we apply the proposed Lyapunov method and compare the effectiveness with formal methods.

Remark 30. *For this special case, the purpose that we treat μ as a time-varied signal with a Lipschitz continuity restriction is to prevent unnecessary extra pumping. The system (16) can be transformed to fit in the general form of (1). Now we provide two ways of transformation.*

(a) *The system (16) is equivalent as*

$$\frac{d}{dt}\begin{bmatrix}\Phi(t)\\ \Psi(t)\\ \mu(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{l_c}(\psi_c - \Psi(t))\\ \frac{1}{16 l_c}\left(\Phi(t) - \mu(t)\sqrt{\Psi(t)}\right)\\ 0\end{bmatrix} + \begin{bmatrix}v(t)\\ 0\\ u_\mu(t)\end{bmatrix}, \tag{17}$$

where $u_\mu : \mathbb{R}_{\geq 0} \to [-0.01, 0.01]$ is an extra input signal on μ such that μ satisfies $|\mu(t+\tau) - \mu(t)| \leq 0.01\tau$. The μ becomes time invariant when $u_\mu \equiv 0$. Equation (17) is in the form of (1), in particular, $u(t) = [v(t), 0, u_\mu(t)]^T$, and

$$f(x) = \begin{bmatrix}\frac{1}{l_c}(\psi_c - \Psi)\\ \frac{1}{16 l_c}\left(\Phi - \mu\sqrt{\Psi}\right)\\ 0\end{bmatrix}, \; g(x) = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The Problem 29 is converted to synthesize v and u_μ such that the trajectory satisfies the reach-avoid-stay specification $(\tilde{W}, \tilde{U}, \tilde{\Omega})$, where $\tilde{W} = \{(\Phi_e(\mu(0)), \Psi_e(\mu(0)), \mu(0)) : \mu(0) \in [0.62, 0.66]\}$; $\tilde{U} = U \times (\mathbb{R} \setminus [0.5, 1])$; $\tilde{\Omega} = \Omega \times [0.5, 1]$.

(b) *The system (16) can be rewritten as*

$$\frac{d}{dt}\begin{bmatrix}\Phi(t)\\ \Psi(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{l_c}(\psi_c - \Psi(t))\\ \frac{1}{16 l_c}\Phi(t)\end{bmatrix} + \begin{bmatrix}v(t)\\ -\mu(t)\sqrt{\Psi(t)}\end{bmatrix}. \tag{18}$$

Let $u(t) = [v(t), \mu(t)]^T$ and

$$g(x(t)) = \begin{bmatrix}1 & 0\\ 0 & -\sqrt{\Psi(t)}\end{bmatrix},$$

then (18) fits in the general form of (1), except that we require additional initial restriction and Lipschitz continuity restriction on μ. We make a little abuse of notation here to define $\mathcal{U} = [-0.05, 0.05] \times [0.5, 0.1]$.

B. Lyapunov method

The Lyapunov method can deal with both forms of the conversion (i.e. (17) and (18)), we only provide an example based on (18). We first derive the sufficient conditions on the state-dependent control signal $u := [v, \mu]^T$ and then embed such conditions as constraints into a quadratic programming framework (QP) [2]. Meanwhile the cost function is selected in a sense that the control effort

$$\|u\|^2 + \frac{2u}{l_c}(\psi_c - \Psi) + \left(\frac{1}{4 l_c}(\Phi - \mu(t))\sqrt{\Psi}\right)^2 \tag{19}$$

is minimized for every $t > 0$.

To derive the sufficient conditions, we need to first select a closed set A such that $A + \varepsilon\mathcal{B} \subseteq \Omega$ and control Lyapunov-barrier functions (V, B) such that the reach-avoid-stay control problem can be reduced to the stabilization with safety guarantee control problem (W, U, A). To demonstrate the sufficiency of the conditions, we simply choose $A = \{\omega\}$. Now we let $h_1(x) = -\|x - \omega\|_\infty + r$ and $h_2(x) = \|x - (0.500, 0.653)\|_\infty - r$, then the sets $\Omega = \{x : h_1(x) \geq 0\}$, $U = \{x : h_2(x) < 0\}$. Therefore, $B(x) = 1/h_2(x)$ is a proper control barrier function as required. The set D can be considered as an open set $\{x : h_2(x) > 0\}$, and $V(x)$ can be chosen as $V(x) = \|x - \omega\|^2$ for all $x \in D \backslash A$. The control strategy for the reach-avoid-stay problem based on (V, B) is then obtained as $\kappa(x)$ (i.e. determined straightforwardly from Theorem 22). However, for Problem 29, we have additional restrictions on μ (i.e. $u([2])$), which is given as

$$\begin{aligned}\mathcal{M} := \{\mu \in [0.5, 1] : \mu(0) \in [0.62, 0.66],\\ |\mu(t+\tau) - \mu(t)| \leq 0.01\tau \quad \forall \tau > 0\}.\end{aligned} \tag{20}$$

Therefore, the final sufficient conditions on the state-dependent control signals are

$$u(t) \in \kappa(x_u(t)) \cap \mathcal{M}. \tag{21}$$

To embed the conditions of (21) into the quadratic programming with the selected cost function, we choose sampling time as 0.1 and use numerical iteration method to obtain the discrete dynamics. The results justified that the sufficient conditions are effective for any μ_0 and $(\Phi_0, \Psi_0) \in W$, but we only show the case when $(\Phi_0, \Psi_0) = (0.5343, 0.6553)$ and $\mu_0 = 0.66$ as an example.

As a result, the control signal u and μ are shown in Figure 3. The sufficient conditions on the signals generated by control Lyapunov-barrier functions are shown to be effectively embedded within the QP with the minimum input energy (19). In particular, the extra conditions on the changing rate of signals are reactively included. The phase portraits of the resulting trajectory is shown in Figure 4.

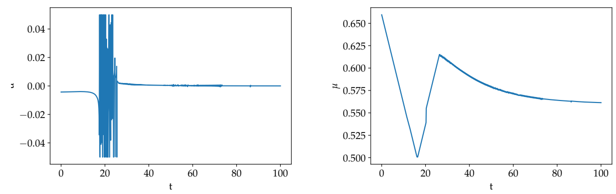

Fig. 3. Control signal u and μ solved by the quadratic programming with condition (21) as constraints and (19) as the cost function.

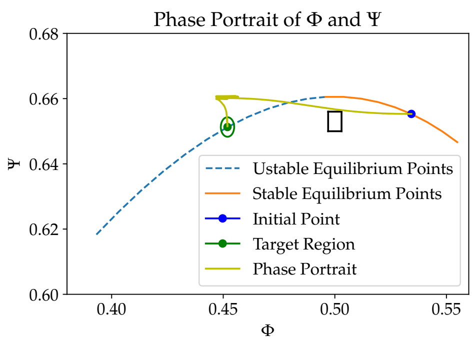

Fig. 4. Phase portrait of (Φ, Ψ) generated based on signal u and μ.

The local Lipschitz continuity of u can also be guaranteed in this framework ([2, Theorem 3]). The synthesis of u and μ mainly depends on the sufficient conditions, as long as it is feasible $(\kappa(x_u(t)) \cap \mathcal{M} \neq \emptyset)$ for the current iteration, it will proceed to the next iteration. The chattering effect of u around time 20 is due to the relatively fast change of μ, which in turn affects the varying speed of the dynamics. The signal μ decided by the QP tends to converge to the $\mu(\omega)$, which is around 0.56.

In particular, when μ is around the Hopf-bifurcation point, the control strategy $\kappa(x)$ can force the trajectory to reach the set A with an exponential rate, the transient speed of the local dynamic will not affect the decision process of u and μ.

C. Discussions on comparisons with formal methods

To apply standard formal methods to Problem 29, we fix a sampling time 0.1 and use ROCS [16] to compute an inner approximation of the winning set w.r.t. the reach-avoid-stay specification as well as synthesize the control strategy. It turned out that the approximated winning set, which is shown in Figure 5, fails to cover the given initial condition. The main reason is that, for μ around the bifurcation point, the sampling time 0.1 is too small to avoid spurious self-transitions in the abstraction. Hence, the reachable set computed on the abstraction is much smaller than the real

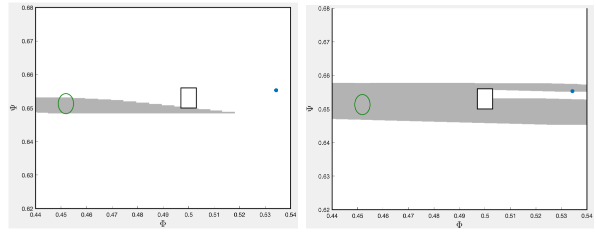

Fig. 5. The approximated winning sets (shaded area) for system (17) (left) with sampling time 0.1 and system (18) (right) with sampling time 0.01, respectively. The green ball: the target set; the black box: the avoid area; the blue dot: the initial condition.

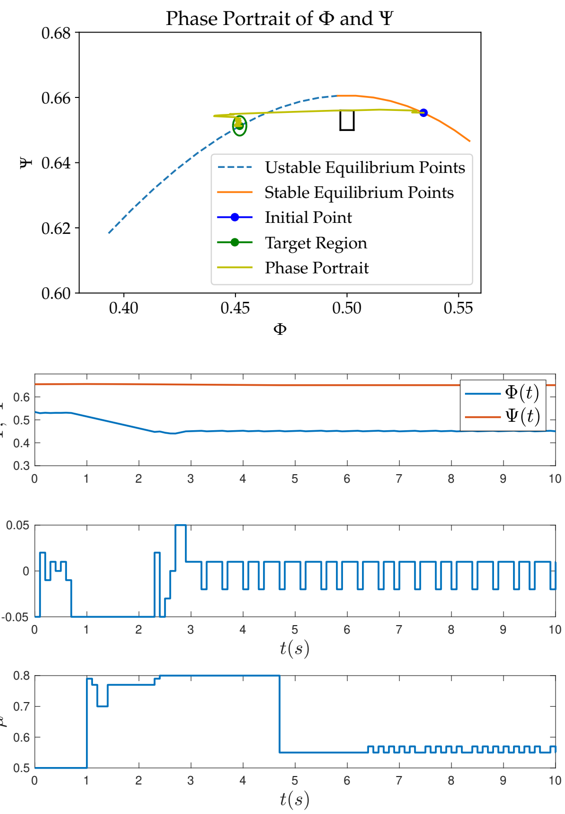

Fig. 6. The closed-loop simulation with the control strategy generated without the constraint on the change rate of μ.

one. The system state variable Ψ evolves very slowly when μ is around the bifurcation point, and this range of μ cannot be avoided due to the constraint on the change rate of μ.

To further see the effect of such a constraint, we remove the constraint $|\mu(t+\tau)-\mu(t)| \leq 0.01\tau$ and perform control synthesis with sampling time 0.01 by using ROCS for (16) directly with μ as a control variable. In this case, a winning set that can cover the initial condition is obtained (see Figure 5). From the closed-loop simulation result given in Figure 6, we notice that the bifurcation point (≈ 0.613) is skipped, leading to potentially unphysical control signals.

In summary, this case study poses challenges to formal methods for two reasons. First, the rate of change for the system state is sensitive to changes in the parameter. This would require the use of a parameter-dependent sampling time in constructing abstractions or computing reachable sets. Current tools [24], [16] and even multiscale methods [10], [13] cannot be readily used to handle such situations. Second, the system includes constraints on the change rates of control inputs. Even though (16) can be reformulated to (17), considering the throttle coefficient μ as a state variable, as opposed to a control variable, will lead to a more conservative control strategy, because of the curse of dimensionality and additional spurious transitions induced by a coarser abstraction/discretization scheme.

VI. CONCLUSIONS

In this paper, we formulated control Lyapunov-barrier functions to develop sufficient conditions for state-dependent control signals w.r.t. the reach-avoid-stay specifications. The formulation of control Lyapunov-barrier functions is based on the entire set of initial conditions from which asymptotic stability with a safety constraint can be achieved. The general topological structure of the initial sets, target sets and unsafe sets leaves us more flexibility to design control Lyapunov-barrier functions.

We investigated the effectiveness in a case study of jet engine compressor control problem. The mathematical model (Moore-Greitzer ODE model) depends on a parameter, the decrease of which can force the system undergo a Hopf-bifurcation. In the control problem, we concern the parameter as a time-varied signal to be decided along with the control signal u. It is shown that the sufficient conditions on u and μ generated by Lyapunov method can be flexibly embedded into a quadratic programming framework with a minimum energy cost. In contrast to formal methods, which fail to handle non-uniform speed of dynamics determined by $\mu(t)$ using the existing tool boxes, Lyapunov methods analytically characterize the topological structure on the solutions w.r.t. the reach-avoid-stay specifications without considering local (linearized) dynamics.

For future work, it would be useful to extend the current approach to control systems with perturbations or noises as encountered in practical applications. It would also be interesting to investigate on the completeness of characterization with Lyapunov-like functions. Motivated by the case study and the discussions on alternatives using formal methods, a more comprehensive comparison of formal methods (tailored to such problems) and Lyapunov methods would be of interest.

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